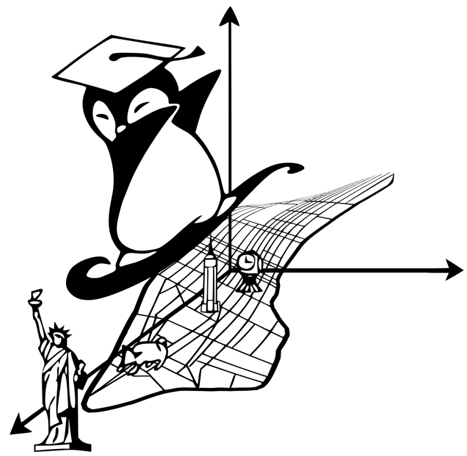
ICMT — Power Round (Division B)



Do not flip open this packet until instructed by your proctor.

For this test, you work in teams to solve multi-part, proof-oriented questions. You have **90 minutes** to complete this round. Questions that use the words "compute," "classify," "find," "draw," "give an example," or "write" require only an answer; no explanation or proof is needed.

For computational questions, please $\boxed{\text{box}}$ your final answer.

All other questions, including those that say "show" or "prove," require proofs. Partial credit may be available for proof-based questions; partial reasoning will not receive credit for computational questions.

Answers should be written on sheets of blank paper, clearly labeled. For each page, write on the **top-right corner**:

1. Your Team Name and Team ID (e.g. Gödel Gerbils, Team #067)
2. Page number out of the total number of pages submitted. (e.g. P2/14)
3. 'Q' + Question number (e.g. Q1.4)*

* *If you have multiple pages for a question, number them and write the total number of pages for the question (e.g. 1/Q2.4, 2/Q2.4 if you are writing the solution of Q2.4 over two pages).*

Only write on ONE side of each sheet of paper.

Only submit **one set** of solutions for the team. Do not turn in any scratch work. After the test, place the sheets you want graded in your team envelope, ordering them by question number from first to last. All sheets should be facing the same side up.

The difficulties of the questions are generally indicated by the point values assigned to them. In your solution for a given question, you may cite the statements of earlier questions (but not later ones) without additional justification, even if you haven't solved them.

Only writing utensils and erasers are permitted. Questions about content clarifications will not be answered by proctors immediately. If a clarification or correction is deemed necessary by the question writing staff, it will be announced to all contestants at the same time.

Good luck!

In this round, we will explore the iconic **Pick's theorem**, which provides us a way to find the area of a simple polygon with integer vertex coordinates. This theorem has become nearly ubiquitous in contest mathematics, but what is not so ubiquitous are its beautiful generalizations and connections to polynomials and harmonic analysis. Along the way, we investigate point-counting and its applications to combinatorics and number theory.

1 From Polygons to Polytopes

1.1 Introducing Pick's Theorem

We will start with some general definitions, which will be useful throughout this power round.

Definition 1.1. A *lattice point* is a point in $\mathbb{R}^d$ where all coordinates are integers. The *integer lattice* $\mathbb{Z}^d$ is the set of all lattice points in d dimensions.

Definition 1.2. For any closed region, $P \subset \mathbb{R}^d$, we define:

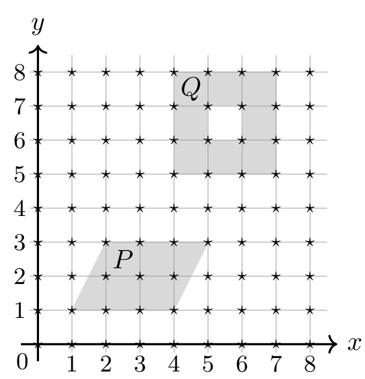
1. $L(P) = \#(P \cap \mathbb{Z}^d)$, the total number of lattice points in the region,
2. $I(P)$, the number of lattice points in the interior of the region (NOT including the boundary), and
3. $B(P)$, the number of lattice points on the boundary of the region.

Observe $L(P) = I(P) + B(P)$.

In this section, we will focus on $d = 2$.

Definition 1.3. A polygon $P \subset \mathbb{R}^2$ is a *lattice polygon* if all its vertices lie in $\mathbb{Z}^2$. A polygon is *simple* if it is non-self-intersecting and has no holes. Intuitively, it has ONE contiguous boundary that separates the polygon into an inside region and an outside region.

Example 1.4. To help visualize the above definitions, here are examples of polygons P and Q that lie in a subset of $\mathbb{R}^2$. We have marked a star at all the lattice points in $\mathbb{Z}^2$. We have shaded in a simple lattice polygon P and a non-simple lattice polygon Q. For these shapes, $L(P) = 11$, $B(P) = 8$, and $I(P) = 3$ while $L(Q) = 16$, $B(Q) = 16$, and $I(Q) = 0$.



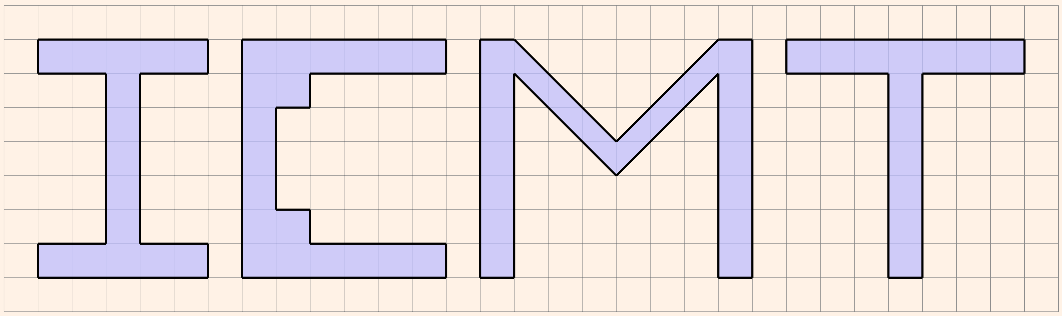
We often want to know the area $A(P)$ of a lattice polygon P. The star of our show, Pick's theorem, allows us to find these areas.

Theorem 1.5. **(Pick's Theorem)** For a simple lattice polygon P,

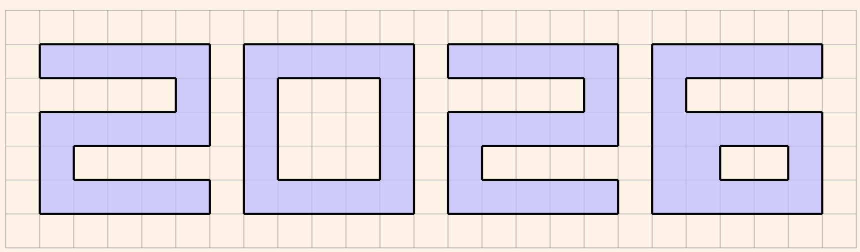
$$A(P) = I(P) + \frac{B(P)}{2} - 1.$$

The grids in the following questions are partitioned into 1×1 squares and represent the 2–D lattice $\mathbb{Z}^2$.

Question 1.1 (10 pts)**.** Compute the number of interior points $I(P)$ and number of boundary points $B(P)$ for each of the four polygons. Compute the sum of the areas of these polygons via Pick's theorem.



Question 1.2 (10 pts)**.** Compute the number of interior points $I(P)$ and number of boundary points $B(P)$, including the polygons that contain holes, for each of the four polygons. Classify each polygon as simple or non-simple. Compute the sum of the areas of these polygons via Pick's theorem.



1.2 A Formal Look at Polygons and Pick's Theorem

To generalize Pick's theorem, we will study how lattice points of regions change as they are scaled up or down. If we scale a region in $\mathbb{R}^2$ by scale factor t, we roughly expect the interior lattice points to scale as proportionally to t^2 and the boundary lattice points to scale as proportionally to t. We show that this is the polynomial growth that actually occurs!

Definition 1.6. Let $P \subset \mathbb{R}^d$ and let $t \in \mathbb{Z}_{\geq 0}$. Define the *t-fold dilation*

$$tP := \{(tx_1, tx_2, \ldots, tx_d) : (x_1, x_2, \ldots, x_d) \in P\}.$$

For convenience, when we work with a dilation factor t as an input, we write $L_P(t) := L(tP) = \#(tP \cap \mathbb{Z}^d)$, $I_P(t) := I(tP)$ and $B_P(t) := B(tP)$.

Let $R = [0, a] \times [0, b]$ with $a, b \in \mathbb{Z}_{>0}$.

Question 1.3 (10 pts)**.** Compute $L(R) = \#(R \cap \mathbb{Z}^2)$, $B(R)$, and $I(R)$ in terms of a, b.

Question 1.4 (10 pts)**.** Compute $L_R(t) = \#(tR \cap \mathbb{Z}^2)$ explicitly as a polynomial in $t \in \mathbb{Z}_{\geq 0}$.

Question 1.5 (10 pts)**.** Find an example of a nonconvex simple lattice polygon and verify Pick's theorem for your example.

Think back to the "2026" diagram and ask yourself, for non-simple polygons, can you directly use the stated Pick's theorem?

In this exercise, let us consider a simpler case of a polygon with "holes":

$$L = [0, a] \times [0, b] \setminus (c_1, c_2) \times (d_1, d_2)$$

where $0 < c_1 < c_2 < a$ and $0 < d_1 < d_2 < b$ are integers.

Question 1.6 (10 pts). Compute $A(L)$, $I(L)$, and $B(L)$. Show that $A = I + \frac{B}{2} - 1$ fails for L and, instead,

$$A(L) = I(L) + \frac{B(L)}{2}.$$

Question 1.7 (5 pts). Find the correct formula for a lattice polygon with H holes.

1.3 A Look at Higher Dimensional Cases

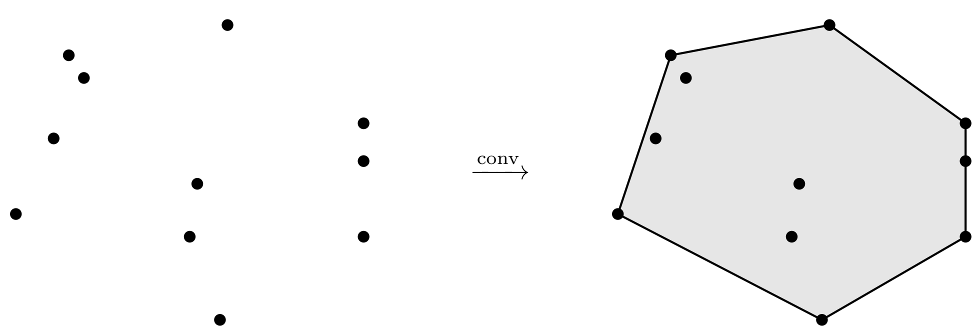
A natural question to ask is whether Pick's theorem generalizes to higher dimensions. For example, can we express the volume of a lattice polytope through some combination of $I(P)$ and $B(P)$? It turns out that higher-dimensional lattice point geometry is a lot richer than its classic two-dimensional counterpart. With this richness comes added complexity.

Definition 1.7. For points $v_1, v_2, \ldots, v_m \in \mathbb{R}^d$, we define their *convex hull* as

$$\operatorname{conv}(\{v_1, v_2, \ldots, v_m\}) := \left\{ \sum_{i=1}^{m} \lambda_i v_i : \lambda_i \geq 0,\ \sum_{i=1}^{m} \lambda_i = 1 \right\}.$$

Intuitively, think of a convex hull as stretching a rubber band around all the points and letting it snap tight.

Example 1.8. The convex hull of the left-hand points is the right-hand shaded region (with the boundary).



Definition 1.9. A region P is a *convex polytope* in $\mathbb{R}^d$ if there exists integer m and points $V = \{v_1, v_2, \ldots, v_m\}$ such that $P = \operatorname{conv}(V)$. The set of *vertices* of P is the minimal subset $V' \subseteq V$ such that $\operatorname{conv}(V') = P$. A polytope is *integral* (or a *lattice polytope*) if all of its vertices lie in $\mathbb{Z}^d$.

Definition 1.10. Points $v_0, v_1, \ldots, v_k \in \mathbb{R}^d$ are *affinely independent* if the vectors $\{v_1 - v_0, v_2 - v_0, \ldots, v_k - v_0\}$ are linearly independent. The *dimension* of a polytope P is the largest integer k such that P contains $k + 1$ affinely independent points.

One can think of polygons as being built up by gluing together a bunch of triangles along their edges. This means studying just triangles yields fruit about other polygons and is thus important. Analogously, we can build up polytopes from building blocks called *simplices*.

Definition 1.11. A convex d–dimensional polytope with exactly $d+1$ vertices $v_0, v_1, \ldots, v_d$ is a *d–simplex*. The *standard d–simplex* is

$$\Delta_d = \operatorname{conv}(\{0, e_1, e_2, \ldots, e_d\}) \subset \mathbb{R}^d,$$

where e_i for $i = 1, 2, \ldots, d$ are standard basis vectors. For any simplex, its (unsigned) volume is

$$\operatorname{Vol}(\Delta) = \frac{1}{d!}\left|\det\left[v_1 - v_0 \mid v_2 - v_0 \mid \cdots \mid v_d - v_0\right]\right|,$$

where the matrix has columns $v_i - v_0$.

For positive integer h, define the tetrahedron

$$T_h = \operatorname{conv}(\{(0,0,0),\ (1,0,0),\ (0,1,0),\ (1,1,h)\}).$$

Question 1.8 (5 pts)**.** Compute $\operatorname{Vol}(T_h)$ using the simplex volume formula.

Question 1.9 (15 pts)**.** Show that for $h \geq 1$, T_h has exactly four boundary lattice points and zero interior lattice points.

Question 1.10 (10 pts)**.** Show that no constants $\alpha, \beta \in \mathbb{R}$ can make $\operatorname{Vol}(P) = I(P) + \alpha B(P) + \beta$ hold for all integral convex 3-dimensional polytopes P.

Question 1.11 (20 pts)**.** Show that any convex d-dimensional polytope has at least $d+1$ vertices.

Question 1.12 (15 pts)**.** Show that for nonnegative integer t,

$$L_{\Delta_d}(t) = \binom{t+d}{d}.$$

2 Ehrhart Theory

2.1 Generating Functions

A generating function is a clever way to package a whole sequence into one expression. By using the language of algebra, we can discover relationships within and between generating functions. These can then translate into identities for certain counting questions. We will exploit these ideas later to simultaneously uncover information about all the values $L_P(0), L_P(1), \ldots$ at once.

Definition 2.1. The *ordinary generating function (OGF)* of a sequence of numbers $\{a_i\}_{i=0}^{\infty} = \{a_0, a_1, \ldots\}$ is defined by

$$\sum_{k=0}^{\infty} a_k x^k.$$

Note that generating functions such as this one are formal power series, i.e., we do not consider any notion of convergence when analyzing the sums of these expressions.

Example 2.2. Consider the sequence $a_i = c^i$, where $c \neq 0$ is a constant, for all $i \geq 0$. Then, for this infinite sequence,

$$\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} c^k x^k = 1 + cx + c^2x^2 + \cdots = \frac{1}{1-cx}.$$

To see this, one can symbolically verify

$$(1-cx)(1+cx+c^2x^2+\dots) = 1.$$

This may remind you of the infinite geometric series formula. If $a_i = c^i$ were truncated to have the first n terms, our OGF would be

$$\sum_{k=0}^{n-1} a_k x^k = \sum_{k=0}^{n-1} c^k x^k = 1 + cx + c^2x^2 + \cdots + c^{n-1}x^{n-1} = \frac{1-(cx)^n}{1-cx}$$

by a similar argument, reminiscent of the finite geometric series formula.

Question 2.1 (5 pts)**.** Find the closed-form OGF for the sequence $a_i = \dfrac{1}{i!}$ for $i \geq 0$. Your answer should not have any summations.

Question 2.2 (10 pts)**.** Recall that a dollar bill is worth 100 cents; pennies, nickels, dimes, and quarters are worth 1, 5, 10, and 25 cents, respectively. Suppose that you were given 100 pennies, 20 nickels, 10 dimes, and 4 quarters (collectively worth 400 cents). Find the OGF that generates the number of ways to make change for n cents with these coins. Express this OGF as a rational function.

Question 2.3 (10 pts)**.** The previous question is a numerical example of the famous *Frobenius Coin Problem*: given coin values $a_1, a_2, \dots, a_d$ (where $\gcd(a_1, \dots, a_d) = 1$), we define $p(t)$, the number of ways to make change for t units of money with as many coins of each type as we want. Find the OGF that gives the number of ways to make change for n units of money for this general case as a rational function.[a]

Question 2.4. Prove that the sequence $\left\{\binom{n}{k}\right\}_{k=0}^{n}$ is generated by the coefficients of the OGF $(1+x)^n$ by:

(a) (5 pts) an algebraic argument.

(b) (10 pts) a combinatorial argument.

Question 2.5. Prove that the sequence $\left\{\binom{n+k-1}{n-1}\right\}_{k=0}^{\infty}$ is generated by the coefficients of the OGF $(1-x)^{-n}$ by:

(a) (10 pts) an algebraic argument.

(b) (10 pts) a combinatorial argument.

[a] The case of $d = 2$ is popularly referred to as the **Chicken McNugget problem** despite Chicken McNuggets originally coming in 3 differently-sized boxes.

2.2 Ehrhart's Theorem

Eugène Ehrhart (1906–2000) did not get his high school diploma until he was 22 and did not begin his PhD until his late 50s. His most impactful contribution was towards lattice point-counting under dilation on general d–polytopes, a field which we call Ehrhart Theory. One key finding in this field is that for a lattice d–polytope P, the function $L_P(t) = \#(tP \cap \mathbb{Z}^d)$ turns out to be a degree–d polynomial in t.

Theorem 2.3. (Ehrhart's Theorem) If P is an integral convex d-polytope, then $L_P(t)$ is a polynomial in $t \in \mathbb{Z}_{\geq 0}$ of degree d.

Theorem 2.4. If $P \subset \mathbb{R}^2$ is an integral convex polygon, then for $t \in \mathbb{Z}_{\geq 0}$,

$$L_P(t) = A(P)\,t^2 + \frac{B(P)}{2}\,t + 1.$$

Let $R = [0, a] \times [0, b]$ with $a, b \in \mathbb{Z}_{>0}$.

Question 2.6 (5 pts). Take the $L_P(t)$ you derived in question 1.4 and match coefficients with $A(R)$ and $B(R)$.

Theorem 2.5. For $\square_d = [0, 1]^d$, we have $L_{\square_d}(t) = (t + 1)^d$.

Question 2.7 (5 pts). Compute $I_{\square_d}(t)$ and show that for $t \geq 1$,

$$B_{\square_d}(t) = (t + 1)^d - (t - 1)^d.$$

2.3 Pyramids

Pyramids provide a way to build a new polytope by stacking copies of an old one. Their lattice-point counts satisfy a simple summation rule, which provides us a way to produce new Ehrhart polynomials from old ones.

Definition 2.6. For $P \subset \mathbb{R}^n$, define the pyramid

$$\mathrm{Pyr}(P) = \mathrm{conv}\Big((P \times \{0\}) \cup \{e_{n+1}\}\Big) \subset \mathbb{R}^{n+1},$$

where e_{n+1} is the $(n + 1)$-th standard basis vector.

Question 2.8 (10 pts). Show that for $m \in \mathbb{Z}_{\geq 0}$,

$$L_{\mathrm{Pyr}(P)}(m) = \sum_{j=0}^{m} L_P(j).$$

Question 2.9 (10 pts). Let $P_d = \mathrm{Pyr}([0, 1]^d)$. Show that

$$L_{P_d}(m) = \sum_{k=1}^{m+1} k^d.$$

2.4 Ehrhart Reciprocity and the h^* Polynomial

Instead of counting lattice points one dilation at a time, we can wrap these values up into a single generating function. This happens to be equivalent to a single rational function, meaning we now need only consider one mathematical object to study every value of $L_P(t)$. We will pay special attention towards the numerator of this rational function, denoted $h^*_P(z)$.

Remark 2.7. In lieu of computing $L_P(t)$ one value at a time, we consider the entire sequence $L_P(0)$, $L_P(1)$, $L_P(2)$, etc. using the generating function

$$\mathrm{Ehr}_P(z) = \sum_{t \geq 0} L_P(t)\, z^t,$$

known as the *Ehrhart series*. One can show that

$$\mathrm{Ehr}_P(z) = \frac{h_P^*(z)}{(1-z)^{d+1}}$$

is always rational, where d is the dimension of P and $h_P^*(z)$ is a polynomial with nonnegative integer coefficients.

Definition 2.8. For a permutation $\pi : \{1, \ldots, d\} \to \{1, \ldots, d\}$, an *ascent* is an index $i \in \{1, \ldots, d-1\}$ with $\pi(i) < \pi(i+1)$. Then, $A(d,k)$ is the number of permutations on a set of size d with exactly $k-1$ ascents.

Question 2.10 (10 pts)**.** Compute $h^*_{\square_d}(z)$ for $\square_d = [0,1]^d$ using

$$h^*_{\square_d}(z) = \sum_{k=1}^{d} A(d,k)\, z^{k-1}$$

for $d = 2$, 3, and 4.

For the following questions, write

$$\mathrm{Ehr}_P(z) = \frac{h_0 + h_1 z + \cdots + h_d z^d}{(1-z)^{d+1}}$$

Question 2.11.

(a) (15 pts) Show that for $t \in \mathbb{Z}_{\geq 0}$

$$L_P(t) = \sum_{j=0}^{d} h_j \binom{t+d-j}{d}.$$

(b) (5 pts) Compute h_0 and h_1.

Question 2.12.

(a) (15 pts) Show that the sum of the h^*–coefficients equals the normalized volume:

$$h_0 + h_1 + \cdots + h_d = d!\,\mathrm{Vol}(P).$$

(b) (5 pts) Verify this for $\square_d = [0,1]^d$.

Question 2.13 (10 pts). Show that for any convex lattice polygon P, question 2.12 implies Pick's theorem.

Question 2.14 (10 pts). Show that $h^*_{\mathrm{Pyr}(P)}(z) = h^*_P(z)$.

Question 2.15. Let $p(t)$ be a polynomial of degree $\le d$ such that

$$\sum_{t\ge 0} p(t)\, z^t \;=\; \frac{h^*(z)}{(1-z)^{d+1}}, \qquad h^*(z) = h_0 + h_1 z + \cdots + h_d z^d.$$

(a) (20 pts) Show that $\deg h^* \le k$ if and only if $p(-1) = p(-2) = \cdots = p(-(d-k)) = 0$.

(b) (15 pts) If $\deg h^* = k$, prove that $p(-(d-k+1)) = (-1)^d h_k \ne 0$.

2.5 Ehrhart–Macdonald Reciprocity

As $L_P(t)$ is a polynomial, it makes sense to evaluate it at negative integers. Reciprocity says these negative values encode lattice points in the interior of P, up to a sign. Hence, the number of interior points come "for free" from the same polynomial.

Definition 2.9. For a polytope $P \subset \mathbb{R}^d$, let P° denote its interior and define

$$L_{P^\circ}(t) := \#(tP^\circ \cap \mathbb{Z}^d) \qquad (t \in \mathbb{Z}_{\ge 1}).$$

Theorem 2.10. (Ehrhart–Macdonald Reciprocity) If P is a convex lattice d-polytope, then for all $t \in \mathbb{Z}_{\ge 1}$,

$$L_P(-t) = (-1)^d\, L_{P^\circ}(t).$$

Let $T_{a,b}$ be the triangle with vertices $(0,0)$, $(a,0)$, and $(0,b)$, where $a, b \in \mathbb{Z}_{>0}$.

Question 2.16 (10 pts). Let $g = \gcd(a,b)$. Show that the hypotenuse contains exactly $g+1$ lattice points, and hence

$$B(T_{a,b}) = a + b + g.$$

Question 2.17 (10 pts). Use Pick's theorem to show

$$L_{T_{a,b}}(t) = \frac{ab}{2}t^2 + \frac{a+b+g}{2}t + 1.$$

Question 2.18 (10 pts). Use reciprocity to express $L_{T^\circ_{a,b}}(1)$ in terms of the coefficients of $L_{T_{a,b}}(t)$.

Question 2.19 (15 pts). Prove the identity

$$\gcd(a,b) = 2\sum_{k=1}^{b-1} \left\lfloor \frac{ka}{b} \right\rfloor + a + b - ab.$$

2.6 The Sawtooth Function

In the previous section, we found that the number of lattice points in a right triangle $T_{a,b}$ depended on $\gcd(a,b)$ and a sum involving the floor function $\lfloor x \rfloor$. It turns out that point-counting arguments can help us prove other number-theoretic identities about more exotic "floor-like" functions.

Definition 2.11. The *sawtooth function* $((x))$ is defined as:

$$((x)) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \in \mathbb{R} \setminus \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z} \end{cases}$$

Question 2.20 (5 pts)**.** Draw the graph of $((x))$. Show that it is an odd function (i.e., $((-x)) = -((x))$) and periodic with period 1.

Definition 2.12. For coprime integers a and b (with $b > 0$), the *Dedekind sum* $s(a, b)$ is defined as:

$$s(a,b) = \sum_{k=1}^{b-1} \left(\left(\frac{k}{b}\right)\right)\left(\left(\frac{ka}{b}\right)\right).$$

These sums might look abstract, but they are intricately related to lattice-point counting.

Question 2.21 (10 pts)**.** Let $P(a,b) = \sum_{k=1}^{b-1} \left(\frac{k}{b} - \frac{1}{2}\right)\left(\frac{ka}{b} - \frac{1}{2}\right)$. Using the definition of the sawtooth function, show that

$$s(a,b) = P(a,b) - \frac{1}{b}\sum_{k=1}^{b-1} k \left\lfloor \frac{ka}{b} \right\rfloor + \frac{1}{2}\sum_{k=1}^{b-1} \left\lfloor \frac{ka}{b} \right\rfloor.$$

Here, s captures geometric information. Notice that the rightmost term is exactly the number of lattice points in the right triangle $T_{b,a}$ and the second term is the related to the x-coordinate of the center of mass of those lattice points.

Question 2.22 (20 pts)**.** Given that $\gcd(a, b) = 1$, the polynomial sum evaluates to

$$P(a,b) = \frac{a(b-1)(b-2)}{12b}$$

and the geometric interpretation of the sums $s(a, b)$, prove the reciprocity identity

$$s(a,b) + s(b,a) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a}\right).$$

The true power of this *Dedekind reciprocity law* lies in its versatility; it serves as a bridge to deep identities across disparate fields of mathematics. In Power A, the students explored its combinatorial roots via the Frobenius Coin Problem from question 2.3. Beyond these counting problems, the theory intersects with linear algebra and complex analysis to reveal even deeper structures through the language of Fourier Analysis. Ultimately, Ehrhart and Dedekind theories stretch far beyond the scope of these pages—related questions remain at the very forefront of modern research. We hope you enjoyed the journey!