Intercollegiate Math Tournament — Power Round (Division B)

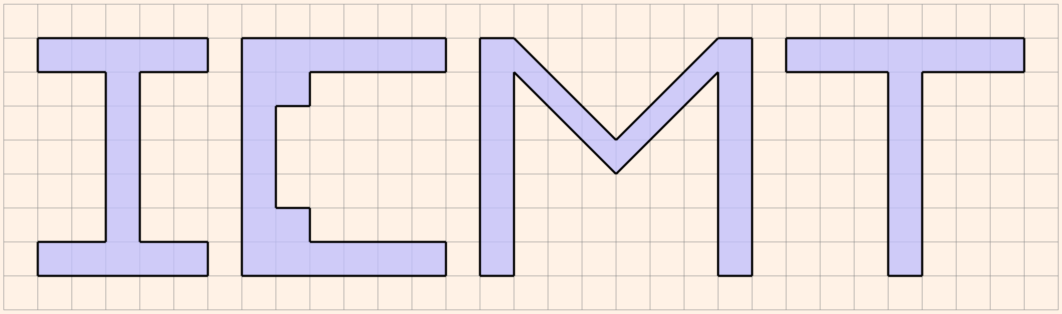
Official Solutions

1 From Polygons to Polytopes

1.1 Introducing Pick's Theorem

The grids in the following questions are partitioned into 1×1 squares and represent the 2–D lattice $\mathbb{Z}^2$.

Question 1.1 (10 pts)**.** Compute the number of interior points $I(P)$ and number of boundary points $B(P)$ for each of the four polygons. Compute the sum of the areas of these polygons via Pick's theorem.



Solution 1.1. For the "I" shape,

$$\boxed{I(P_{\text{I}}) = 0}, \qquad \boxed{B(P_{\text{I}}) = 32}, \qquad A(P_{\text{I}}) = I(P_{\text{I}}) + \frac{B(P_{\text{I}})}{2} - 1 = 0 + \frac{32}{2} - 1 = 15.$$

For the "C" shape,

$$\boxed{I(P_{\text{C}}) = 2}, \qquad \boxed{B(P_{\text{C}}) = 36}, \qquad A(P_{\text{C}}) = I(P_{\text{C}}) + \frac{B(P_{\text{C}})}{2} - 1 = 2 + \frac{36}{2} - 1 = 19.$$

For the "M" shape,

$$\boxed{I(P_{\text{M}}) = 0}, \qquad \boxed{B(P_{\text{M}}) = 42}, \qquad A(P_{\text{M}}) = I(P_{\text{M}}) + \frac{B(P_{\text{M}})}{2} - 1 = 0 + \frac{42}{2} - 1 = 20.$$

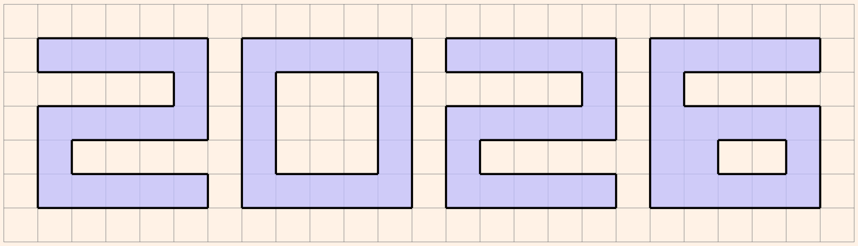
For the "T" shape,

$$\boxed{I(P_{\text{T}}) = 0}, \qquad \boxed{B(P_{\text{T}}) = 28}, \qquad A(P_{\text{T}}) = I(P_{\text{T}}) + \frac{B(P_{\text{T}})}{2} - 1 = 0 + \frac{28}{2} - 1 = 13.$$

The sum of all of these areas is

$$15 + 19 + 20 + 13 = \boxed{67}.$$

Question 1.2 (10 pts)**.** Compute the number of interior points $I(P)$ and number of boundary points $B(P)$, including the polygons that contain holes, for each of the four polygons. Classify each polygon as simple or non-simple. Compute the sum of the areas of these polygons via Pick's theorem.



Solution 1.2. For the "2" shape, there are no holes; hence it is $\boxed{\text{simple}}$. In this case, we use Pick's theorem normally. Thus,

$$\boxed{I(P_2) = 0}, \qquad \boxed{B(P_2) = 36}, \qquad A(P_2) = I(P_2) + \frac{B(P_2)}{2} - 1 = 0 + \frac{36}{2} - 1 = 17.$$

For the "0" shape, there are holes; hence it is $\boxed{\text{non-simple}}$. Notice that both the entire "0" shape with its corresponding hole are simple shapes. We find the area of the entire region and subtract that from the area of the hole. Thus,

$$\boxed{I(P_{0,\text{total}}) = 16}, \qquad \boxed{B(P_{0,\text{total}}) = 20}, \qquad A(P_{0,\text{total}}) = I(P_{0,\text{total}}) + \frac{B(P_{0,\text{total}})}{2} - 1 = 16 + \frac{20}{2} - 1 = 25$$

and

$$\boxed{I(P_{0,\text{hole}}) = 4}, \qquad \boxed{B(P_{0,\text{hole}}) = 12}, \qquad A(P_{0,\text{hole}}) = I(P_{0,\text{hole}}) + \frac{B(P_{0,\text{hole}})}{2} - 1 = 4 + \frac{12}{2} - 1 = 9.$$

The area of the "0" shape with the hole is

$$A(P_0) = A(P_{0,\text{total}}) - A(P_{0,\text{hole}}) = 25 - 9 = 16.$$

For the "6" shape, there are holes; hence it is $\boxed{\text{non-simple}}$. Notice that both the entire "6" shape with its corresponding hole are simple shapes. We find the area of the entire region and subtract that from the area of the hole. Thus,

$$\boxed{I(P_{6,\text{total}}) = 8}, \qquad \boxed{B(P_{6,\text{total}}) = 28}, \qquad A(P_{6,\text{total}}) = I(P_{6,\text{total}}) + \frac{B(P_{6,\text{total}})}{2} - 1 = 8 + \frac{28}{2} - 1 = 21$$

and

$$\boxed{I(P_{6,\text{hole}}) = 0}, \qquad \boxed{B(P_{6,\text{hole}}) = 6}, \qquad A(P_{6,\text{hole}}) = I(P_{6,\text{hole}}) + \frac{B(P_{6,\text{hole}})}{2} - 1 = 0 + \frac{6}{2} - 1 = 2.$$

The area of the "6" shape with the hole is

$$A(P_6) = A(P_{6,\text{total}}) - A(P_{6,\text{hole}}) = 21 - 2 = 19.$$

The sum of all of these areas is

$$17 + 16 + 17 + 19 = \boxed{69}.$$

1.2 A Formal Look at Polygons and Pick's Theorem

Let $R = [0, a] \times [0, b] \subset \mathbb{R}^2$ with $a, b \in \mathbb{Z}_{>0}$.

Question 1.3 (10 pts). Compute $L(R)$, $B(R)$, and $I(R)$ in terms of a and b.

Solution 1.3. With a grid of $(a+1) \times (b+1)$ points, the number of boundary points is calculated by counting the number of points on its perimeter and subtracting this from the number of points on the corners. We get

$$\boxed{L(R) = (a+1)(b+1)}, \quad \boxed{B(R) = 2(a+b)}, \quad \boxed{I(R) = (a-1)(b-1)}.$$

Question 1.4 (10 pts). Compute $L_R(t)$ explicitly as a polynomial in $t \in \mathbb{Z}_{\geq 0}$.

Solution 1.4. We apply $L(R)$ from the previous question to the region $[0, ta] \times [0, tb]$. We get

$$\boxed{L_R(t) = ab\,t^2 + (a+b)t + 1}.$$

Question 1.5 (10 pts). Find an example of a nonconvex simple lattice polygon and verify Pick's theorem for your example.

Solution 1.5. Answers may vary. One possible example is an L–shape with coordinates $\{(0,0), (2,0), (2,1), (1,1), (1,2), (0,2)\}$. Counting the number of boundary and interior points as well as finding its area via usual techniques gives $\boxed{A = 3}$, $\boxed{B = 8}$, and $\boxed{I = 0}$. Checking with Pick's theorem, we have $A = 3 = 0 + \frac{8}{2} - 1$.

In this exercise, let us consider a simpler case of a polygon with "holes":

$$L = [0, a] \times [0, b] \setminus (c_1, c_2) \times (d_1, d_2)$$

where $0 < c_1 < c_2 < a$ and $0 < d_1 < d_2 < b$ are integers.

Question 1.6 (10 pts). Compute $A(L)$, $I(L)$, and $B(L)$. Show that $A = I + \frac{B}{2} - 1$ fails for L and, instead,

$$A(L) = I(L) + \frac{B(L)}{2}.$$

Solution 1.6. Let $w = c_2 - c_1$ and $h = d_2 - d_1$. Then,

$$\boxed{A(L) = ab - wh}, \quad \boxed{B(L) = 2(a+b) + 2(w+h)}, \quad \boxed{I(L) = (a-1)(b-1) - (w+1)(h+1)}.$$

We have that

$$\begin{aligned} I(L) + \frac{B(L)}{2} &= (a-1)(b-1) - (c_2 - c_1 + 1)(d_2 - d_1 + 1) + a + b + (c_2 - c_1) + (d_2 - d_1) \\ &= (ab + 1) - ((c_2 - c_1)(d_2 - d_1) + 1) \\ I(L) + \frac{B(L)}{2} &= ab - (c_2 - c_1)(d_2 - d_1). \end{aligned}$$

Solution 1.6. (cont.) This means that the original Pick's formula could not have been correct, and that the new one (that provides a correction factor for holes) is. We see that

$$I(L) + \tfrac{B(L)}{2} - 1 = ab - wh - 1$$
$$I(L) + \tfrac{B(L)}{2} - 1 = A(L) - 1$$
$$A(L) = I(L) + \frac{B(L)}{2},$$

as desired. □

Question 1.7 (5 pts). Find the correct formula for a lattice polygon with H holes.

Solution 1.7. For H holes, $\boxed{A(P) = I(P) + \frac{B(P)}{2} - 1 + H}$.

1.3 A Look at Higher Dimensional Cases

For positive integer h, define the tetrahedron

$$T_h = \operatorname{conv}(\{(0,0,0),\ (1,0,0),\ (0,1,0),\ (1,1,h)\}).$$

Question 1.8 (5 pts). Compute $\operatorname{Vol}(T_h)$ using the simplex volume formula.

Solution 1.8. By the simplex volume formula, we have

$$\det\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & h\end{pmatrix} = h \implies \boxed{\operatorname{Vol}(T_h) = \frac{h}{6}}.$$

Question 1.9 (15 pts). Show that for $h \geq 1$, T_h has exactly four boundary lattice points and zero interior lattice points.

Solution 1.9. Writing $(x, y, z) \in T_h$ as a convex combination gives

$$a(0,0,0) + b(1,0,0) + c(0,1,0) + d(1,1,h) \text{ with } a + b + c + d = 1,\ a, b, c, d \geq 0.$$

Then $x = b+d$, $y = c+d$, and $z = hd$ are our parameterized coordinates. Observe $a \geq 0$ implies $b+c+d \leq 1$. Thus, we have $x = b + d \leq 1$ and $y = c + d \leq 1$. For lattice points, each $x, y \in \{0, 1\}$. We consider the following two cases.

1. Cases $x = 0$ or $y = 0$: Then, assert $z = 0$, giving us vertices $\{(0,0,0), (1,0,0), (0,1,0)\}$.
2. Case $x = y = 1$: From $b+d = c+d = 1$ we get $b = c = 1-d$. Then $a = 1-b-c-d = 1-2(1-d)-d = d-1$. For $a \geq 0$, we need $d \geq 1$. Since $d \leq 1$ from both $d \leq 1 - b \leq 1$ and $d \leq 1 - c \leq 1$, we have $d = 1$, so $z = h$. Thus $(1, 1, z) \in T_h \Leftrightarrow z = h$.

Therefore, $T_h \cap \mathbb{Z}^3 = \{(0,0,0), (1,0,0), (0,1,0), (1,1,h)\}$ with $B(T_h) = 4$ and $I(T_h) = 0$, as desired. □

Question 1.10 (10 pts)**.** Show that no constants $\alpha, \beta \in \mathbb{R}$ can make $\mathrm{Vol}(P) = I(P) + \alpha B(P) + \beta$ hold for all integral convex 3-polytopes P.

Solution 1.10. We argue by contradiction. Suppose $\mathrm{Vol}(P) = I(P) + \alpha B(P) + \beta$ for some constants $\alpha, \beta \in \mathbb{R}$ holds for all integral convex 3-polytopes P. Then, it must hold for a specific polytope P, say, the *Reeve tetrahedron* T_h defined for Questions 1.8 and 1.9. For this polytope T_h, we have $h/6 = 0 + 4\alpha + \beta$ for all $h \geq 1$. However, the left hand side of this equation depends on h while the right hand side is constant, a contradiction. ☐

Question 1.11 (20 pts)**.** Show that any convex d-dimensional polytope has at least $d+1$ vertices.

Solution 1.11. By definition, a d-dimensional polytope must contain $d+1$ affinely independent points. If some polytope P has vertices $v_0, v_1, \cdots, v_k$ for $k < d$, all points in P may be written as $v_0 + \mathbf{x} \cdot \mathbf{v}$, where $\mathbf{x} \in [0,1]^k$ and $\mathbf{v} = (v_1 - v_0, v_2 - v_0, \cdots, v_k - v_0)$. Therefore, we can write this basis of all points in P of $k+1 < d+1$ points, meaning the dimension is at most $k < d$. So P must have at least $d+1$ vertices to be d-dimensional. ☐

Question 1.12 (15 pts)**.** Show that for nonnegative integer t,

$$L_{\Delta_d}(t) = \binom{t+d}{d}.$$

Solution 1.12. We have $t\Delta_d = \{x \in \mathbb{R}^d : x_i \geq 0,\ \sum x_i \leq t\}$. For $(x_1, x_2, \ldots, x_d) \in t\Delta_d \cap \mathbb{Z}^d$, we must satisfy $x_i \geq 0$ for all coordinates x_i. Since every point $(x_1, x_2, \ldots, x_d)$ is a convex combination of vertices of $t\Delta_d$, we require the sum of the coordinates of each point $x \in \mathbb{R}^d$ to be at most the maximum sum of the coordinates of some vertex, which is t. Since we have d integers that sum to at most t, by stars and bars and the hockey stick identity, there are $L_{\Delta_d}(t) = \binom{t+d}{d}$ total points, as desired. ☐

2 Ehrhart Theory

2.1 Generating Functions

Question 2.1 (5 pts)**.** Find the closed-form OGF for the sequence $a_i = \dfrac{1}{i!}$ for $i \geq 0$. Your answer should not have any summations.

Solution 2.1. By plugging in the sequence $a_i = \frac{1}{i!}$, $i \geq 0$ into the OGF form, we have

$$\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \boxed{e^x}.$$

Question 2.2 (10 pts)**.** Recall that a dollar bill is worth 100 cents; pennies, nickels, dimes, and quarters are worth 1, 5, 10, and 25 cents, respectively. Suppose that you were given 100 pennies, 20 nickels, 10 dimes, and 4 quarters (collectively worth 400 cents). Find the OGF that generates the number of ways to make change for n cents with these coins. Express this OGF as a rational function.

Solution 2.2. Since there are finitely many coins given for each denomination, we write down each OGF individually for each denomination. The number of ways to make change for n units of money given only 100 pennies is

$$1 + x + x^2 + \cdots + x^{99} + x^{100} = \frac{1 - x^{101}}{1 - x}.$$

The number of ways to make change for n units of money given only 20 nickels is

$$1 + x^5 + x^{10} + \cdots + x^{95} + x^{100} = \frac{1 - (x^5)^{21}}{1 - x^5} = \frac{1 - x^{105}}{1 - x^5}.$$

The number of ways to make change for n units of money given only 10 dimes is

$$1 + x^{10} + x^{20} + \cdots + x^{90} + x^{100} = \frac{1 - (x^{10})^{11}}{1 - x^{10}} = \frac{1 - x^{110}}{1 - x^{10}}.$$

The number of ways to make change for n units of money given only 4 quarters is

$$1 + x^{25} + x^{50} + x^{75} + x^{100} = \frac{1 - (x^{25})^5}{1 - x^{25}} = \frac{1 - x^{125}}{1 - x^{25}}.$$

Therefore, the OGF that generates the number of ways to make change for n cents with these coins is

$$\boxed{\frac{1 - x^{101}}{1 - x} \cdot \frac{1 - x^{105}}{1 - x^5} \cdot \frac{1 - x^{110}}{1 - x^{10}} \cdot \frac{1 - x^{125}}{1 - x^{25}}}$$

as a rational function.

Question 2.3 (10 pts)**.** The previous question is a numerical example of the famous *Frobenius Coin Problem*: given coin values $a_1, a_2, \ldots, a_d$ (where $\gcd(a_1, \ldots, a_d) = 1$), we define $p(t)$, the number of ways to make change for t units of money with as many coins of each type as we want. Find the OGF that gives the number of ways to make change for n units of money for this general case as a rational function.[a]

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[a] The case of $d = 2$ is popularly referred to as the **Chicken McNugget problem** despite Chicken McNuggets originally coming in 3 differently-sized boxes.

Solution 2.3. Since there are infinitely many coins given for each denomination, we write down each OGF individually for each denomination. The number of ways to make change for n units of money given only a denomination of a_i units of money, $1 \le i \le d$ is $1 + x^{a_i} + x^{2a_i} + \cdots = \frac{1}{1 - x^{a_i}}$. Therefore, the OGF that generates the number of ways to make change for n units of money with these coins is

$$\boxed{\frac{1}{(1 - x^{a_1})(1 - x^{a_2}) \cdots (1 - x^{a_{d-1}})(1 - x^{a_d})} = \prod_{i=1}^{d} \frac{1}{1 - x^{a_i}}}.$$

Question 2.4. Prove that the sequence $\left\{\binom{n}{k}\right\}_{k=0}^{n}$ is generated by the coefficients of the OGF $(1 + x)^n$ by:

(a) (5 pts) an algebraic argument.

(b) (10 pts) a combinatorial argument.

Solution 2.4. (a) **(1st Solution)** By the binomial theorem,

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} (x)^k (1)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} x^k. \quad \square$$

(2nd Solution) We use mathematical induction on n.

Base Case: For the cases $n = 0$, we have $(1+x)^0 = 1 = \binom{0}{0} x^0$.

Inductive Step: Suppose that $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$. Then,

$$\begin{aligned}
(1+x)^{n+1} &= (1+x) \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{n} \binom{n}{k} x^k + \sum_{k=0}^{n} \binom{n}{k} x^{k+1} = \sum_{k=0}^{n} \binom{n}{k} x^k + \sum_{k=1}^{n+1} \binom{n}{k-1} x^k \\
&= \binom{n}{0} x^0 + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1} \right] x^k + \binom{n}{n} x^{n+1} = 1 + \sum_{k=1}^{n} \binom{n+1}{k} x^k + x^{n+1} \\
(1+x)^{n+1} &= \sum_{k=0}^{n+1} \binom{n+1}{k} x^k,
\end{aligned}$$

where Pascal's identity was used to sum the two binomial coefficients. $\square$

(3rd Solution) Let $f(x) = (1+x)^n$. Then,

$$f'(x) = n(1+x)^{n-1} \qquad f''(x) = n(n-1)(1+x)^{n-2} \qquad \cdots \qquad f^{(k)}(x) = n(n-1)\cdots(n-k+1)(1+x)^{n-k}$$

and, by Taylor's theorem,

$$f(x) = (1+x)^n = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{n(n-1)\cdots(n-k+1)}{k!} x^k = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} x^k = \sum_{k=0}^{n} \binom{n}{k} x^k. \quad \square$$

(b) The binomial coefficient $\binom{n}{k}$ equals the number of ways that k can be written as an unordered sum of n integers $a_1, a_2, \ldots, a_n$, where $a_i \in \{0,1\}$ for each i. Since each $a_i \in \{0,1\}$, we can express the two choices for each a_i as $x^0 + x^1 = 1 + x$. Since there are n integers, the resulting OGF is $(1+x)^n$. $\square$

Question 2.5. Prove that the sequence $\left\{\binom{n+k-1}{n-1}\right\}_{k=0}^{\infty}$ is generated by the coefficients of the OGF $(1-x)^{-n}$ by:

(a) (10 pts) an algebraic argument.

(b) (10 pts) a combinatorial argument.

Solution 2.5. (a) **(1st Solution)** By the extended binomial theorem,

$$(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} (-x)^k (1)^{n-k} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} (-x)^k = \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} x^k. \quad \square$$

(2nd Solution) We use mathematical induction on n.

Base Case: For the case $n = 1$, we have

$$(1-x)^{-1} = \frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{k=0}^{\infty} \binom{k}{0} x^k = \sum_{k=0}^{\infty} \binom{1+k-1}{1-1} x^k.$$

Inductive Step: Suppose that $(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} x^k$. By Pascal's identity, $\binom{n+k-1}{n-1} + \binom{n+k-1}{n} = \binom{n+k}{n}$. Therefore,

$$\begin{aligned}
(1-x)^{-n} &= \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} x^k = \sum_{k=0}^{\infty} \left[\binom{n+k}{n} - \binom{n+k-1}{n} \right] x^k \\
&= \sum_{k=0}^{\infty} \binom{n+k}{n} x^k - \sum_{k=0}^{\infty} \binom{n+k-1}{n} x^k = \sum_{k=0}^{\infty} \binom{n+k}{n} x^k - \sum_{k=0}^{\infty} \binom{n+k}{n} x^{k+1} \\
&= \sum_{k=0}^{\infty} \binom{n+k}{n} x^k (1-x) = (1-x) \sum_{k=0}^{\infty} \binom{n+k}{n} x^k \\
(1-x)^{-(n+1)} &= \sum_{k=0}^{\infty} \binom{n+k}{n} x^k. \quad \square
\end{aligned}$$

(3rd Solution) Let $f(x) = (1-x)^{-n}$. Then,

$$f'(x) = (-n)(1-x)^{-(n+1)}, \qquad f''(x) = n(n+1)(1-x)^{-(n+2)}, \qquad \cdots$$

$$f^{(k)}(x) = (-1)^k n(n+1)\cdots(n+k-1)(1-x)^{-(n+k)}$$

and, by Taylor's theorem,

$$\begin{aligned}
f(x) = (1-x)^{-n} &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} (-x)^k = \sum_{k=0}^{\infty} \frac{(-1)^k n(n+1)\cdots(n+k-1)}{k!} (-x)^k = \sum_{k=0}^{\infty} \frac{(n+k-1)!}{(n-1)!k!} x^k \\
(1-x)^{-n} &= \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} x^k. \quad \square
\end{aligned}$$

(b) The binomial coefficient $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$ equals the number of ways that k can be written as an ordered sum of n nonnegative integers $a_1, a_2, \ldots, a_n$. Since a_i is a nonnegative integer for each i, we can express the choices for each a_i as $x^0 + x^1 + x^2 + \cdots = 1 + x + x^2 + \cdots = \frac{1}{1-x}$. Since there are n nonnegative integers, the resulting OGF is $\left(\frac{1}{1-x}\right)^n = (1-x)^{-n}$. $\square$

2.2 Ehrhart's Theorem

Let $R = [0,a] \times [0,b]$ with $a, b \in \mathbb{Z}_{>0}$.

Question 2.6 (5 pts). Take the $L_R(t)$ you derived in question 1.4 and match coefficients with $A(R)$ and $B(R)$.

Solution 2.6. We match coefficients of $L_R(t) = ab\,t^2 + (a+b)t + 1$ (from Question 1.4) with this new definition of R.

- Coefficient of t^2: $A(R) = ab = \mathrm{Vol}_2(R)$, the volume of the polygon in two dimensions (area).
- Coefficient of t: $\frac{B(R)}{2} = a+b$, so $B(R) = 2(a+b)$, which matches the lattice point count on its perimeter.
- Constant term: $1 = \chi(R)$, the Euler characteristic for this simple polygon R.

Question 2.7 (5 pts). Compute $I_{\square_d}(t)$ and show that for $t \geq 1$,

$$B_{\square_d}(t) = (t+1)^d - (t-1)^d.$$

Solution 2.7. We have $t\square_d = [0,t]^d$, so $L_{\square_d}(t) = (t+1)^d$. The coordinates of the interior points satisfy $1 \leq x_i \leq t-1$ for each $1 \leq i \leq d$, giving $I(t\square_d) = (t-1)^d$ for $t \geq 1$. Thus, $\boxed{I(t\square_d) = (t-1)^d}$ and $B(t\square_d) = (t+1)^d - (t-1)^d$, as desired. □

2.3 Pyramids

Question 2.8 (10 pts). Show that for $m \in \mathbb{Z}_{\geq 0}$,

$$L_{\mathrm{Pyr}(P)}(m) = \sum_{j=0}^{m} L_P(j).$$

Solution 2.8. A lattice point in $m\,\mathrm{Pyr}(P) \subset \mathbb{R}^{n+1}$ has last coordinate $k \in \{0, 1, \ldots, m\}$. Fix such a k. The cross-section at height k is affinely isomorphic to $(m-k)P$, so it contributes $L_P(m-k)$ lattice points. Summing over all k and reindexing with $j = m-k$ gives

$$L_{\mathrm{Pyr}(P)}(m) = \sum_{j=0}^{m} L_P(j)\,,$$

as desired. □

Question 2.9 (10 pts). Let $P_d = \mathrm{Pyr}([0,1]^d)$. Show that

$$L_{P_d}(m) = \sum_{k=1}^{m+1} k^d.$$

Solution 2.9. For $P = [0,1]^d$, we have $L_P(j) = (j+1)^d$, hence

$$L_{P_d}(m) = \sum_{j=0}^{m} (j+1)^d = \sum_{k=1}^{m+1} k^d\,,$$

as desired. □

2.4 Ehrhart Reciprocity and the h^* Polynomial

Question 2.10 (10 pts). Compute $h^*_{\square_d}(z)$ for $\square_d = [0,1]^d$ using

$$h^*_{\square_d}(z) = \sum_{k=1}^{d} A(d,k)\, z^{k-1}$$

for $d = 2$, 3, and 4.

Solution 2.10. The h^*-polynomial of the unit cube $\square_d = [0,1]^d$ is given by $h^*_{\square_d}(z) = \sum_{k=1}^{d} A(d,k)\, z^{k-1}$, where $A(d,k)$ is the Eulerian number counting permutations of $\{1,2,\dots,d\}$ with $k-1$ ascents.

- For $d = 2$, the Eulerian numbers are $A(2,1) = 1$, $A(2,2) = 1$, and

$$\boxed{h^*_{\square_2}(z) = 1 + z}.$$

- For $d = 3$, the Eulerian numbers are $A(3,1) = 1$, $A(3,2) = 4$, $A(3,3) = 1$, and

$$\boxed{h^*_{\square_3}(z) = 1 + 4z + z^2}.$$

- For $d = 4$, the Eulerian numbers are $A(4,1) = 1$, $A(4,2) = 11$, $A(4,3) = 11$, $A(4,4) = 1$, and

$$\boxed{h^*_{\square_4}(z) = 1 + 11z + 11z^2 + z^3}.$$

For the following questions, write

$$\mathrm{Ehr}_P(z) = \frac{h_0 + h_1 z + \cdots + h_d z^d}{(1-z)^{d+1}}$$

Question 2.11.

(a) (15 pts) Show that for $t \in \mathbb{Z}_{\geq 0}$,

$$L_P(t) = \sum_{j=0}^{d} h_j \binom{t+d-j}{d}.$$

(b) (5 pts) Compute h_0 and h_1.

Solution 2.11. (a) The $(1-z)^{d+1}$ factor in the denominator encourages us to use the result of Question 2.5. We obtain

$$\sum_{t\geq 0} L_P(t)\, z^t = \left(\sum_{j=0}^{d} h_j z^j\right)\left(\sum_{m\geq 0}\binom{m+d}{d} z^m\right) = \sum_{j=0}^{d} h_j \left(\sum_{m\geq 0}\binom{m+d}{d} z^{m+j}\right).$$

Extracting the coefficient of z^t by selecting $m = t-j$, we obtain

$$L_P(t) = \sum_{j=0}^{d} h_j \cdot \left[\text{coefficient of } z^{t-j} \text{ in } \tfrac{1}{(1-z)^{d+1}}\right] = \sum_{j=0}^{d} h_j \binom{t-j+d}{d} = \sum_{j=0}^{d} h_j \binom{t+d-j}{d},$$

as desired. ☐

(b)
- Setting $t = 0$, we have $L_P(0) = \sum_{j=0}^{d} h_j \binom{d-j}{d}$. Since $\binom{d-j}{d} = 0$ for $j \geq 1$, we get $L_P(0) = h_0$. However, $0P = \{(0,0,\dots,0)\}$ has exactly one lattice point, so $\boxed{h_0 = 1}$.
- Setting $t = 1$, we have $L_P(1) = h_0\binom{d+1}{d} + h_1\binom{d}{d} + \sum_{j\geq 2} h_j \binom{d+1-j}{d}$. For $j \geq 2$, $d+1-j < d$ so $\binom{d+1-j}{d} = 0$. Thus $L_P(1) = (d+1) + h_1$, giving

$$\boxed{h_1 = L_P(1) - (d+1)}.$$

Since $L_P(1) = \#(P \cap \mathbb{Z}^d) = I(P) + B(P)$, we have $\boxed{h_1 = I(P) + B(P) - d - 1}$.

Question 2.12.

(a) (15 pts) Show that the sum of the h^*–coefficients equals the normalized volume

$$h_0 + h_1 + \cdots + h_d = d!\,\mathrm{Vol}(P).$$

(b) (5 pts) Verify part (a) for $\Box_d = [0,1]^d$.

Solution 2.12. (a) From part (a) of Question 2.11, $L_P(t) = \displaystyle\sum_{j=0}^{d} h_j \binom{t+d-j}{d}$. As a polynomial in t, the leading term of $\binom{t+d-j}{d}$ is $\frac{t^d}{d!}$, independent of j. Therefore the leading coefficient of $L_P(t)$ is

$$[t^d]\, L_P(t) = \frac{1}{d!} \sum_{j=0}^{d} h_j.$$

By Ehrhart's theorem, the leading coefficient of $L_P(t)$ equals $\mathrm{Vol}(P)$. Hence

$$\mathrm{Vol}(P) = \frac{1}{d!} \sum_{j=0}^{d} h_j \quad \Longrightarrow \quad h_0 + h_1 + \cdots + h_d = d!\, \mathrm{Vol}(P),$$

as desired. ☐

(b) For $\square_d = [0,1]^d$, $\mathrm{Vol}(\square_d) = 1$, so $h_0 + h_1 + \cdots + h_d = d!$. Indeed,

- for $d = 2$, $h^*(z) = 1 + z$ with a coefficient sum of $1 + 1 = 2 = 2!$;
- for $d = 3$, $h^*(z) = 1 + 4z + z^2$ with a coefficient sum of sum $1 + 4 + 1 = 6 = 3!$; and
- for $d = 4$, $h^*(z) = 1 + 11z + 11z^2 + z^3$ with a coefficient sum of sum $1 + 11 + 11 + 1 = 24 = 4!$.

You can check also with the identity $\displaystyle\sum_{k=1}^{d} A(d,k) = d!$.

Question 2.13 (10 pts)**.** Show that for any convex lattice polygon P, question 2.12 implies Pick's theorem.

Solution 2.13. Let P be a convex lattice polygon (so $d = 2$). From Question 2.11(a),

$$\begin{aligned} L_P(t) &= h_0 \binom{t+2}{2} + h_1 \binom{t+1}{2} + h_2 \binom{t}{2} \\ &= h_0 \frac{(t+2)(t+1)}{2} + h_1 \frac{t(t+1)}{2} + h_2 \frac{t(t-1)}{2} \\ L_P(t) &= \frac{h_0 + h_1 + h_2}{2}\, t^2 + \frac{3h_0 + h_1 - h_2}{2}\, t + h_0 \end{aligned}$$

By Ehrhart's theorem for $d = 2$, $L_P(t) = A(P)\, t^2 + \frac{B(P)}{2}\, t + 1$. By matching the leading coefficients, we have

$$A(P) = \frac{h_0 + h_1 + h_2}{2} \implies h_0 + h_1 + h_2 = 2A(P) = 2!A(P),$$

which matches the desired form from Question 2.12(a). Evaluating $L_P(t)$ at $t = 1$, we have

$$L_P(1) = A(P) + \frac{B(P)}{2} + 1 = I(P) + B(P),$$

where $L_P(1) = I(P) + B(P)$ counts all lattice points in P. Rearranging this equation, we get

$$A(P) = I(P) + \frac{B(P)}{2} - 1,$$

which is Pick's theorem. ☐

Question 2.14 (10 pts). Show that $h^*_{\mathrm{Pyr}(P)}(z) = h^*_P(z)$.

Solution 2.14. Let $P \subset \mathbb{R}^n$ be a d-dimensional lattice polytope. By Question 2.8, $L_{\mathrm{Pyr}(P)}(m) = \sum_{j=0}^{m} L_P(j)$.

The Ehrhart series of $\mathrm{Pyr}(P)$ is

$$\sum_{m\geq 0} L_{\mathrm{Pyr}(P)}(m)\, z^m = \sum_{m\geq 0} \left(\sum_{j=0}^{m} L_P(j) \right) z^m = \frac{1}{1-z} \sum_{j\geq 0} L_P(j)\, z^j,$$

since multiplying a generating function by $\frac{1}{1-z}$ computes partial sums of the coefficients. Thus,

$$\sum_{m\geq 0} L_{\mathrm{Pyr}(P)}(m)\, z^m = \frac{1}{1-z} \cdot \mathrm{Ehr}_P(z) = \frac{1}{1-z} \cdot \frac{h^*_P(z)}{(1-z)^{d+1}} = \frac{h^*_P(z)}{(1-z)^{d+2}}.$$

Since $\mathrm{Pyr}(P)$ is $(d+1)$-dimensional, its Ehrhart series has the form $\frac{h^*_{\mathrm{Pyr}(P)}(z)}{(1-z)^{(d+1)+1}}$. Comparing this polynomial with $\mathrm{Ehr}_P(z)$ with dimension $d+1$ gives $h^*_{\mathrm{Pyr}(P)}(z) = h^*_P(z)$, as desired. $\square$

Question 2.15. Let $p(t)$ be a polynomial of degree $\leq d$ such that

$$\sum_{t\geq 0} p(t)\, z^t \;=\; \frac{h^*(z)}{(1-z)^{d+1}}, \text{ where } h^*(z) = h_0 + h_1 z + \cdots + h_d z^d.$$

(a) (20 pts) Show that $\deg h^* \leq k$ if and only if $p(-1) = p(-2) = \cdots = p(-(d-k)) = 0$.

(b) (15 pts) If $\deg h^* = k$, prove that $p(-(d-k+1)) = (-1)^d h_k \neq 0$.

Solution 2.15. (a) We use the formula $L_P(t) = p(t) = \sum_{j=0}^{d} h_j \binom{t+d-j}{d}$ from Question 2.11(a). For negative integers $t = -m$ with $m \geq 1$,

$$p(-m) = \sum_{j=0}^{d} h_j \binom{d-j-m}{d}.$$

Now $\binom{d-j-m}{d} = 0$ whenever $0 \leq d-j-m < d$, i.e., when $1 \leq j+m \leq d$. For $j+m > d$, we use the identity $\binom{n}{d} = (-1)^d \binom{d-1-n}{d}$ for $n < 0$ (extended binomial theorem), giving $\binom{d-j-m}{d} = (-1)^d \binom{j+m-1}{d}$. Therefore, for $1 \leq m \leq d$:

$$p(-m) = (-1)^d \sum_{j=d-m+1}^{d} h_j \binom{j+m-1}{d}. \tag{1}$$

(⇒) If $\deg h^* \leq k$, then $h_j = 0$ for all $j > k$. For $m \in \{1, 2, \ldots, d-k\}$, the sum in (1) runs over $j > d-m \geq d-(d-k) = k$, so every h_j in the sum vanishes. Hence $p(-m) = 0$ for each $m \in \{1, 2, \ldots, d-k\}$.

(⇐) Suppose $p(-1) = p(-2) = \cdots = p(-(d-k)) = 0$. We prove $h_d = h_{d-1} = \cdots = h_{k+1} = 0$ by downward induction on m.

- *Base Case*:
  - For the case $m = 1$, $p(-1) = (-1)^d h_d \binom{d}{d} = (-1)^d h_d = 0$, so $h_d = 0$.
  - For the case $m = 2$, $p(-2) = (-1)^d \left[h_{d-1}\binom{d}{d} + h_d \binom{d+1}{d}\right] = (-1)^d h_{d-1} = 0$ (using $h_d = 0$), so $h_{d-1} = 0$.
- *Inductive Step*: At step $m = \ell$, $h_d = h_{d-1} = \cdots = h_{d-\ell+2} = 0$, so $p(-\ell) = (-1)^d h_{d-\ell+1} \cdot 1 = 0$, giving $h_{d-\ell+1} = 0$. At $m = d-k$, we obtain $h_{k+1} = 0$.

Thus $h_{k+1} = h_{k+2} = \cdots = h_d = 0$, i.e. $\deg h^* \leq k$. □

(b) If $\deg h^* = k$, then $h_k \neq 0$ and $h_{k+1} = h_{k+2} = \cdots = h_d = 0$. Setting $m = d-k+1$ in (1):

$$p(-(d-k+1)) = (-1)^d \sum_{j=k}^{d} h_j \binom{j+d-k}{d}.$$

Since $h_{k+1} = h_{k+2} = \cdots = h_d = 0$, only the $j = k$ term survives, leaving us with

$$p(-(d-k+1)) = (-1)^d h_k \binom{k+d-k}{d} = (-1)^d h_k \binom{d}{d} = (-1)^d h_k.$$

Since $\deg h^* = k$, we have $h_k \neq 0$, so

$$p(-(d-k+1)) = (-1)^d \, h_k \binom{d}{d} = (-1)^d h_k \neq 0,$$

as desired. □

2.5 Ehrhart-Macdonald Reciprocity

Let $T_{a,b}$ be the triangle with vertices $(0,0)$, $(a,0)$, and $(0,b)$, where $a, b \in \mathbb{Z}_{>0}$.

Question 2.16 (10 pts)**.** Let $g = \gcd(a,b)$. Show that the hypotenuse contains exactly $g+1$ lattice points, and hence

$$B(T_{a,b}) = a + b + g.$$

Solution 2.16. Let $g = \gcd(a,b)$. The hypotenuse is the segment from $(a,0)$ to $(0,b)$, whose direction vector is $\langle -a, b\rangle$. For this line segment, it contains $\gcd(a,b)+1 = g+1$ lattice points. The other two sides contribute $(a+1)$ points on the x-axis and $(b+1)$ points on the y-axis. Adding these quantities and subtracting the 3 corner vertices counted twice gives

$$B(T_{a,b}) = (a+1)+(b+1)+(g+1)-3 = a+b+g,$$

as desired. ☐

Question 2.17 (10 pts)**.** Use Pick's theorem to show

$$L_{T_{a,b}}(t) = \frac{ab}{2}t^2 + \frac{a+b+g}{2}t + 1.$$

Solution 2.17. The area of this triangle is $A(T_{a,b}) = \frac{ab}{2}$ and, from Question 2.16, $B(T_{a,b}) = a+b+g$. Therefore, by Ehrhart's theorem for $d = 2$, we have

$$L_{T_{a,b}}(t) = \frac{ab}{2}t^2 + \frac{a+b+g}{2}t + 1.$$

as desired. ☐

Question 2.18 (10 pts)**.** Use reciprocity to express $L_{T^\circ_{a,b}}(1)$ in terms of the coefficients of $L_{T_{a,b}}(t)$.

Solution 2.18. For $d = 2$, reciprocity says $L_P(-t) = L_{P^\circ}(t)$, so

$$L_{T^\circ_{a,b}}(1) = L_{T_{a,b}}(-1) = \frac{ab}{2} - \frac{a+b+g}{2} + 1 = \boxed{\frac{ab-a-b-g+2}{2}}.$$

Question 2.19 (15 pts)**.** Prove the identity

$$\gcd(a,b) = 2\sum_{k=1}^{b-1}\left\lfloor \frac{ka}{b}\right\rfloor + a + b - ab.$$

Solution 2.19. We count the total number of interior lattice points of $T_{a,b}$ by counting up the number of interior points from each horizontal line $y = k$ for $k = 1, 2, \ldots, b-1$. The hypotenuse of $T_{a,b}$ is defined by the equation $\frac{x}{a} + \frac{y}{b} = 1$, i.e. $x = a\left(1 - \frac{k}{b}\right)$. The x-coordinates of the interior points of $T_{a,b}$ satisfy $0 < x < a\left(1 - \frac{k}{b}\right)$, so at a particular height $y = k$, the number of interior points is

$$\#\left\{x \in \mathbb{Z} : 1 \leq x < a\left(1 - \frac{k}{b}\right)\right\} = \left\lfloor \frac{a(b-k)-1}{b} \right\rfloor = a - 1 - \left\lfloor \frac{ak}{b} \right\rfloor.$$

Summing over $k = 1, 2, \ldots, b-1$ gives

$$L_{T^\circ_{a,b}}(1) = \sum_{k=1}^{b-1}\left(a - 1 - \left\lfloor \frac{ak}{b} \right\rfloor\right) = (a-1)(b-1) - \sum_{k=1}^{b-1}\left\lfloor \frac{ak}{b} \right\rfloor.$$

Since $L_{T^\circ_{a,b}}(1) = \frac{ab-a-b-g+2}{2}$ from Question 2.18, solving for $g = \gcd(a,b)$ gives

$$2\sum_{k=1}^{b-1}\left\lfloor \frac{ak}{b} \right\rfloor = ab - a - b + g \quad \Longleftrightarrow \quad g = 2\sum_{k=1}^{b-1}\left\lfloor \frac{ak}{b} \right\rfloor + a + b - ab,$$

as desired. □

2.6 The Sawtooth Function

Question 2.20 (5 pts)**.** Draw the graph of $((x))$. Show that it is an odd function (i.e., $((-x)) = -((x))$) and periodic with period 1.

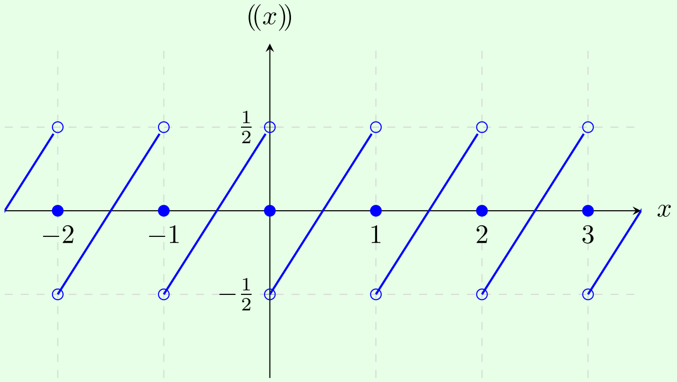
Solution 2.20. First, we show periodicity. For any $x \in \mathbb{R} \setminus \mathbb{Z}$, we see that

$$((x+1)) = (x+1) - \lfloor x+1 \rfloor - \frac{1}{2} = (x+1) - (\lfloor x \rfloor + 1) - \frac{1}{2} = x - \lfloor x \rfloor - \frac{1}{2} = ((x)).$$

For $x \in \mathbb{Z}$, $((x+1)) = 0 = ((x))$. Thus, it is periodic with period 1. Next, we show it is an odd function. For $x \in \mathbb{Z}$, $((-x)) = 0 = -0 = -((x))$. For $x \in \mathbb{R} \setminus \mathbb{Z}$, we use the property that $\lfloor -x \rfloor = -\lfloor x \rfloor - 1$ to see that

$$((-x)) = -x - \lfloor -x \rfloor - \frac{1}{2} = -x - (-\lfloor x \rfloor - 1) - \frac{1}{2} = -x + \lfloor x \rfloor + \frac{1}{2} = -\left(x - \lfloor x \rfloor - \frac{1}{2}\right) = -((x)). \quad \square$$

The graph of $((x))$ is shown below.



Question 2.21 (10 pts). Let $P(a,b) = \sum_{k=1}^{b-1}\left(\frac{k}{b}-\frac{1}{2}\right)\left(\frac{ka}{b}-\frac{1}{2}\right)$. Using the definition of the sawtooth function, show that

$$s(a,b) = P(a,b) - \frac{1}{b}\sum_{k=1}^{b-1} k\left\lfloor \frac{ka}{b}\right\rfloor + \frac{1}{2}\sum_{k=1}^{b-1}\left\lfloor \frac{ka}{b}\right\rfloor.$$

Solution 2.21. By definition, $s(a,b) = \sum_{k=1}^{b-1}\left(\left(\frac{k}{b}\right)\right)\left(\left(\frac{ka}{b}\right)\right)$. Because $0 < 1 \le k \le b-1 < b$, we have $0 < \frac{k}{b} < 1$, which means $\lfloor \frac{k}{b} \rfloor = 0$. Thus, $\left(\left(\frac{k}{b}\right)\right) = \frac{k}{b} - \frac{1}{2}$. We see that

$$\begin{aligned}
s(a,b) &= \sum_{k=1}^{b-1}\left(\frac{k}{b}-\frac{1}{2}\right)\left(\frac{ka}{b}-\left\lfloor\frac{ka}{b}\right\rfloor-\frac{1}{2}\right)\\
&= \sum_{k=1}^{b-1}\left(\frac{k}{b}-\frac{1}{2}\right)\left(\frac{ka}{b}-\frac{1}{2}\right) - \sum_{k=1}^{b-1}\left(\frac{k}{b}-\frac{1}{2}\right)\left\lfloor\frac{ka}{b}\right\rfloor\\
s(a,b) &= P(a,b) - \sum_{k=1}^{b-1}\frac{k}{b}\left\lfloor\frac{ka}{b}\right\rfloor + \frac{1}{2}\sum_{k=1}^{b-1}\left\lfloor\frac{ka}{b}\right\rfloor,
\end{aligned}$$

as desired. □

Question 2.22 (20 pts). Given that the polynomial sum evaluates to $P(a,b) = \frac{a(b-1)(b-2)}{12b}$ and the geometric interpretation of the sums $s(a,b)$, prove the reciprocity identity

$$s(a,b) + s(b,a) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a}\right).$$

Solution 2.22. We use the expression for $s(a,b)$ derived in Question 2.21 to obtain

$$\begin{aligned}
s(a,b)+s(b,a) &= P(a,b) - \frac{1}{b}\sum_{k=1}^{b-1}k\left\lfloor\frac{ka}{b}\right\rfloor + \frac{1}{2}\sum_{k=1}^{b-1}\left\lfloor\frac{ka}{b}\right\rfloor + P(b,a) - \frac{1}{a}\sum_{k=1}^{a-1}k\left\lfloor\frac{kb}{a}\right\rfloor + \frac{1}{2}\sum_{k=1}^{a-1}\left\lfloor\frac{kb}{a}\right\rfloor\\
&= \left[P(a,b)+P(b,a)\right] - \left[\frac{1}{b}\sum_{k=1}^{b-1}k\left\lfloor\frac{ka}{b}\right\rfloor + \frac{1}{a}\sum_{k=1}^{a-1}k\left\lfloor\frac{kb}{a}\right\rfloor\right] + \left[\frac{1}{2}\sum_{k=1}^{b-1}\left\lfloor\frac{ka}{b}\right\rfloor + \frac{1}{2}\sum_{k=1}^{a-1}\left\lfloor\frac{kb}{a}\right\rfloor\right]\\
s(a,b)+s(b,a) &= I_1 - I_2 + I_3.
\end{aligned}$$

Using the given expression for $P(a,b)$ gives

$$I_1 = P(a,b) + P(b,a) = \frac{a(b-1)(b-2)}{12b} + \frac{b(a-1)(a-2)}{12a}.$$

Using Question 2.19 with $g = \gcd(a,b) = 1$ on the rightmost term of $s(a,b)+s(b,a)$ gives

$$I_3 = \frac{1}{2}\sum_{k=1}^{b-1}\left\lfloor\frac{ka}{b}\right\rfloor + \frac{1}{2}\sum_{k=1}^{a-1}\left\lfloor\frac{kb}{a}\right\rfloor = \frac{(a-1)(b-1)}{2}.$$

Solution 2.22. (cont.) We now focus on I_2, the middle term of $s(a,b)+s(b,a)$, for the rest of the required computations. For this term, we cannot consider both sections individually. For arbitrary a, b such that $\gcd(a,b)=1$, we want to geometrically evaluate the sum

$$aS_1+bS_2,\ \text{where } S_1=\sum_{k=1}^{b-1}k\left\lfloor\frac{ka}{b}\right\rfloor \text{ and } S_2=\sum_{k=1}^{a-1}k\left\lfloor\frac{kb}{a}\right\rfloor.$$

Let R be the set of interior integer lattice points in the rectangle $(0,b)\times(0,a)$. The diagonal line $y=\frac{a}{b}x$ (or $ax=by$) divides R into a lower triangle T_1 and an upper triangle T_2. Because $\gcd(a,b)=1$, no integer points lie exactly on the diagonal. Notice that the number of lattice points in the k-th column of T_1 is exactly $\lfloor\frac{ka}{b}\rfloor$. Thus, multiplying by the column index k and summing over all columns gives the sum of the x-coordinates of all points in T_1. Therefore, $S_1=\sum_{T_1}x$. By identical logic for the rows of T_2, $S_2=\sum_{T_2}y$. We can now rewrite our target sum over the points in the regions as

$$aS_1+bS_2=\sum_{(x,y)\in T_1}ax+\sum_{(x,y)\in T_2}by.$$

For any point in T_1, $ax>by$, meaning $\max(ax,by)=ax$. For any point in T_2, $ax<by$, meaning $\max(ax,by)=by$. Since T_1 and T_2 partition the rectangle R, we can unify the sum as

$$aS_1+bS_2=\sum_{(x,y)\in R}\max(ax,by).$$

Using the algebraic identity $\max(A,B)=\frac{A+B+|A-B|}{2}$, we split the sum over R by

$$\sum_R\max(ax,by)=\frac{1}{2}\sum_R(ax+by)+\frac{1}{2}\sum_R|ax-by|.$$

The first term is a straightforward arithmetic sum over the independent variables x and y.

$$\frac{1}{2}\sum_R(ax+by)=\frac{1}{2}\left(a\sum_{x=1}^{b-1}\sum_{y=1}^{a-1}x+b\sum_{x=1}^{b-1}\sum_{y=1}^{a-1}y\right)=\frac{1}{2}\left(a(a-1)\frac{b(b-1)}{2}+b(b-1)\frac{a(a-1)}{2}\right)=\frac{ab(a-1)(b-1)}{2}$$

For the second term, observe that the map $(x,y)\mapsto(b-x,a-y)$ is a bijection on R that negates the value of $ax-by$. Therefore, the absolute values are perfectly symmetric, and $\frac{1}{2}\sum_R|ax-by|=\sum_{T_1}(ax-by)$. We evaluate this sum column by column over T_1. Then,

$$\sum_{T_1}(ax-by)=\sum_{x=1}^{b-1}\sum_{y=1}^{\lfloor\frac{ax}{b}\rfloor}(ax-by)=\sum_{x=1}^{b-1}\left(ax\left\lfloor\frac{ax}{b}\right\rfloor-\frac{b}{2}\left\lfloor\frac{ax}{b}\right\rfloor\left(\left\lfloor\frac{ax}{b}\right\rfloor+1\right)\right)$$

We substitute the fractional part $\{\frac{ax}{b}\}=\frac{ax}{b}-\lfloor\frac{ax}{b}\rfloor$ to remove the floors. Remarkably, when expanding the squared term, the cross-terms perfectly cancel out the $ax\{\frac{ax}{b}\}$ terms, leaving

$$\sum_{T_1}(ax-by)=\sum_{x=1}^{b-1}\left(\frac{a^2x^2}{2b}-\frac{b}{2}\left\{\frac{ax}{b}\right\}^2-\frac{ax}{2}+\frac{b}{2}\left\{\frac{ax}{b}\right\}\right).$$

Because $\gcd(a,b)=1$, the fractional parts $\{\frac{ax}{b}\}$ exactly permute the values $\frac{1}{b},\frac{2}{b},\dots,\frac{b-1}{b}$ as x ranges from 1 to $b-1$. Substituting standard formulas for the sums of integers and squares yields

$$\sum_{T_1}(ax-by)=\frac{a^2(b-1)(2b-1)}{12}-\frac{(b-1)(2b-1)}{12}-\frac{ab(b-1)}{4}+\frac{b(b-1)}{4}=\frac{(a-1)(b-1)(2ab-a-b-1)}{12}$$

Solution 2.22. (cont.) Plugging the closed forms solutions for these two terms yields

$$\begin{aligned} aS_1 + bS_2 = \sum_R \max(ax, by) &= \frac{1}{2}\sum_R (ax+by) + \frac{1}{2}\sum_R |ax - by| \\ &= \frac{ab(a-1)(b-1)}{2} + \frac{(a-1)(b-1)(2ab-a-b-1)}{12}. \end{aligned}$$

The remaining sum needed is $I_2 = \frac{1}{b}S_1 + \frac{1}{a}S_2 = \frac{1}{ab}(aS_1 + bS_2)$. Thus,

$$\begin{aligned} I_2 &= \frac{1}{ab}\left(\frac{ab(a-1)(b-1)}{2} + \frac{(a-1)(b-1)(2ab-a-b-1)}{12}\right) \\ &= \frac{(a-1)(b-1)}{2} + \frac{(a-1)(b-1)}{12}\left(2 - \frac{1}{b} - \frac{1}{a} - \frac{1}{ab}\right) \\ &= \frac{2(a-1)(b-1)}{3} - \frac{(a-1)(b-1)}{12}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{ab}\right) \\ I_2 &= \frac{2(a-1)(b-1)}{3} - \frac{(a-1)(b-1)}{12}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{ab}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} -I_2 + I_3 &= -\left[\frac{2(a-1)(b-1)}{3} - \frac{(a-1)(b-1)}{12}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{ab}\right)\right] + \frac{(a-1)(b-1)}{2} \\ &= -\frac{(a-1)(b-1)}{6} + \frac{(a-1)(b-1)}{12}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{ab}\right) \end{aligned}$$

and

$$\begin{aligned} s(a,b) + s(b,a) &= I_1 + (-I_2 + I_3) \\ &= \left[\frac{a(b-1)(b-2)}{12b} + \frac{b(a-1)(a-2)}{12a}\right] + \left[-\frac{(a-1)(b-1)}{6} + \frac{(a-1)(b-1)}{12}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{ab}\right)\right] \\ &= \frac{a(b^2-3b+2)}{12b} + \frac{b(a^2-3a+2)}{12a} - \frac{ab-a-b+1}{6} + \frac{(ab-a-b+1)(a+b+1)}{12ab} \\ &= \frac{ab}{12} - \frac{a}{4} + \frac{a}{6b} + \frac{ab}{12} - \frac{b}{4} + \frac{b}{6a} - \frac{ab}{6} + \frac{a}{6} + \frac{b}{6} - \frac{1}{6} + \frac{\left(1 - \frac{1}{a} - \frac{1}{b} + \frac{1}{ab}\right)(a+b+1)}{12} \\ &= \frac{ab}{6} - \frac{a+b}{4} + \frac{a}{6b} + \frac{b}{6a} - \frac{ab}{6} + \frac{a+b}{6} - \frac{1}{6} + \frac{a+b+1-\frac{a}{b}-1-\frac{1}{b}-1-\frac{b}{a}-\frac{1}{a}+\frac{1}{b}+\frac{1}{a}+\frac{1}{ab}}{12} \\ &= -\frac{1}{6} - \frac{a+b}{12} + \frac{a}{6b} + \frac{b}{6a} + \frac{a+b}{12} - \frac{a}{12b} - \frac{b}{12a} + \frac{1}{12ab} - \frac{1}{12} \\ &= -\frac{1}{4} + \frac{a}{12b} + \frac{b}{12a} + \frac{1}{12ab} \\ s(a,b) + s(b,a) &= -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a}\right), \end{aligned}$$

as desired. □