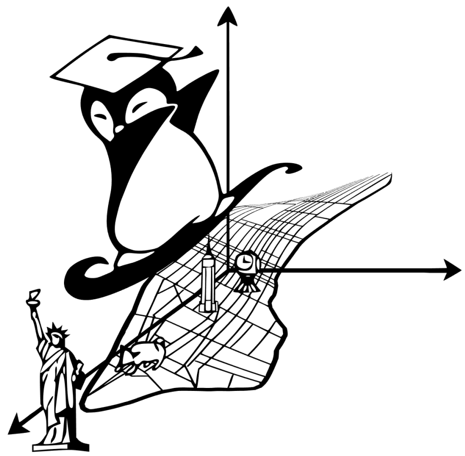
ICMT — Power Round (Division A)



Do not flip open this packet until instructed by your proctor.

For this test, you work in teams to solve multi-part, proof-oriented questions. You have **90 minutes** to complete this round. Questions that use the words "compute," "classify," "find," "draw," "give an example," or "write" require only an answer; no explanation or proof is needed.

For computational questions, please box your final answer.

All other questions, including those that say "show" or "prove," require proofs. Partial credit may be available for proof-based questions; partial reasoning will not receive credit for computational questions.

Answers should be written on sheets of blank paper, clearly labeled. For each page, write on the **top-right corner**:

1. Your Team Name and Team ID (e.g. Gödel Gerbils, Team #067)
2. Page number out of the total number of pages submitted. (e.g. P2/14)
3. 'Q' + Question number (e.g. Q1.4)*

* *If you have multiple pages for a question, number them and write the total number of pages for the question (e.g. 1/Q2.4, 2/Q2.4 if you are writing the solution of Q2.4 over two pages).*

Only write on ONE side of each sheet of paper.

Only submit **one set** of solutions for the team. Do not turn in any scratch work. After the test, place the sheets you want graded in your team envelope, ordering them by question number from first to last. All sheets should be facing the same side up.

The difficulties of the questions are generally indicated by the point values assigned to them. In your solution for a given question, you may cite the statements of earlier questions (but not later ones) without additional justification, even if you haven't solved them.

Only writing utensils and erasers are permitted. Questions about content clarifications will not be answered by proctors immediately. If a clarification or correction is deemed necessary by the question writing staff, it will be announced to all contestants at the same time.

Good luck!

In this round, we will explore the iconic **Pick's theorem**, which provides us a way to find the area of a simple polygon with integer vertex coordinates. This theorem has become nearly ubiquitous in contest mathematics, but what is not so ubiquitous are its beautiful generalizations and connections to polynomials and harmonic analysis. Along the way, we investigate point-counting and its applications to combinatorics and number theory.

1 From Polygons to Polytopes

1.1 Introducing Pick's Theorem

We will start with some general definitions, which will be useful throughout this power round.

Definition 1.1. A *lattice point* is a point in $\mathbb{R}^d$ where all coordinates are integers. The *integer lattice* $\mathbb{Z}^d$ is the set of all lattice points in d dimensions.

Definition 1.2. For any closed region, $P \subset \mathbb{R}^d$, we define:

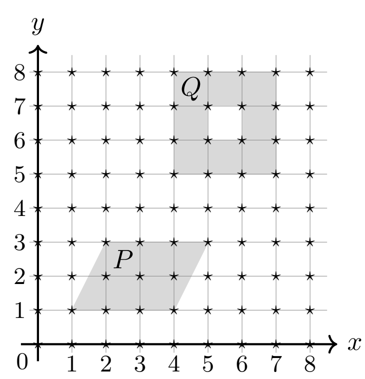
1. $L(P) = \#(P \cap \mathbb{Z}^d)$, the total number of lattice points in the region,
2. $I(P)$, the number of lattice points in the interior of the region (NOT including the boundary), and
3. $B(P)$, the number of lattice points on the boundary of the region.

Observe $L(P) = I(P) + B(P)$.

In this section, we will focus on $d = 2$.

Definition 1.3. A polygon $P \subset \mathbb{R}^2$ is a *lattice polygon* if all its vertices lie in $\mathbb{Z}^2$. A polygon is *simple* if it is non-self-intersecting and has no holes. Intuitively, it has ONE contiguous boundary that separates the polygon into an inside region and an outside region.

Example 1.4. To help visualize the above definitions, here are examples of polygons P and Q that lie in a subset of $\mathbb{R}^2$. We have marked a star at all the lattice points in $\mathbb{Z}^2$. We have shaded in a simple lattice polygon P and a non-simple lattice polygon Q. For these shapes, $L(P) = 11$, $B(P) = 8$, and $I(P) = 3$ while $L(Q) = 16$, $B(Q) = 16$, and $I(Q) = 0$.



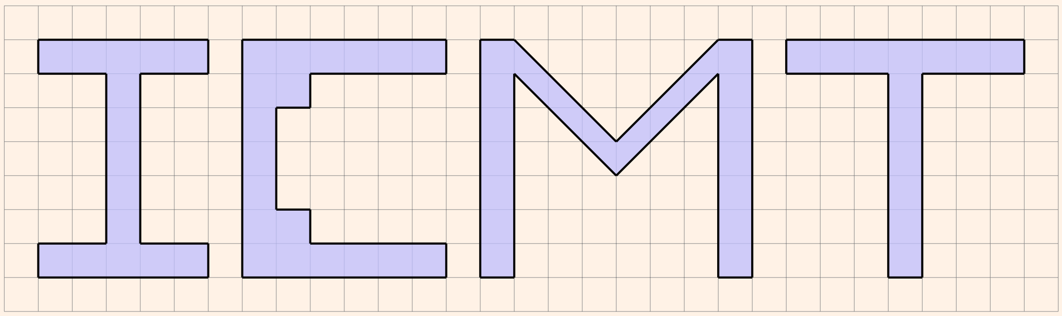
We often want to know the area $A(P)$ of a lattice polygon P. The star of our show, Pick's theorem, allows us to find these areas.

Theorem 1.5. (Pick's Theorem) For a simple lattice polygon P,

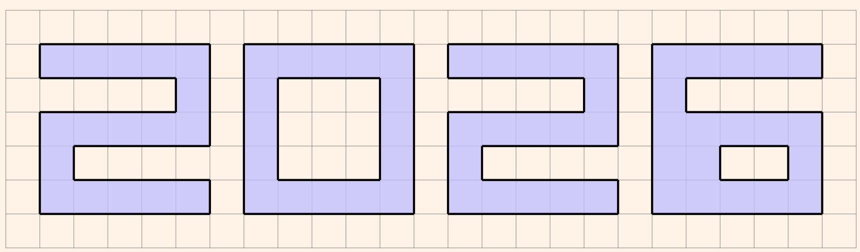
$$A(P) = I(P) + \frac{B(P)}{2} - 1.$$

The grids in the following questions are partitioned into 1×1 squares and represent the 2–D lattice $\mathbb{Z}^2$.

Question 1.1 (10 pts)**.** Compute the number of interior points $I(P)$ and number of boundary points $B(P)$ for each of the four polygons. Compute the sum of the areas of these polygons via Pick's theorem.



Question 1.2 (10 pts)**.** Compute the number of interior points $I(P)$ and number of boundary points $B(P)$, including the polygons that contain holes, for each of the four polygons. Classify each polygon as simple or non-simple. Compute the sum of the areas of these polygons via Pick's theorem.



1.2 A Formal Look at Polygons and Pick's Theorem

To generalize Pick's theorem, we will study how the set of lattice points of regions change as those regions are scaled up or down. If we scale a region in $\mathbb{R}^2$ by a scale factor t, we roughly expect the interior lattice points to scale proportionally to t^2 and the boundary lattice points to scale proportionally to t. We show that this is the polynomial growth that actually occurs!

Definition 1.6. Let $P \subset \mathbb{R}^d$ and $t \in \mathbb{Z}_{\geq 0}$. Define the *t-fold dilation* as

$$tP := \{(tx_1, tx_2, \ldots, tx_d) : (x_1, x_2, \ldots, x_d) \in P\}.$$

For convenience, when we work with a dilation factor t as an input, we write $L_P(t) := L(tP) = \#(tP \cap \mathbb{Z}^d)$, $I_P(t) := I(tP)$, and $B_P(t) := B(tP)$.

Let $R = [0, a] \times [0, b] \subset \mathbb{R}^2$ with $a, b \in \mathbb{Z}_{>0}$.

Question 1.3 (10 pts)**.** Compute $L(R)$, $B(R)$, and $I(R)$ in terms of a and b.

Question 1.4 (10 pts)**.** Compute $L_R(t)$ explicitly as a polynomial in $t \in \mathbb{Z}_{\geq 0}$.

Question 1.5 (10 pts)**.** Find an example of a nonconvex simple lattice polygon and verify Pick's theorem for your example.

1.3 A Look at Higher Dimensional Cases

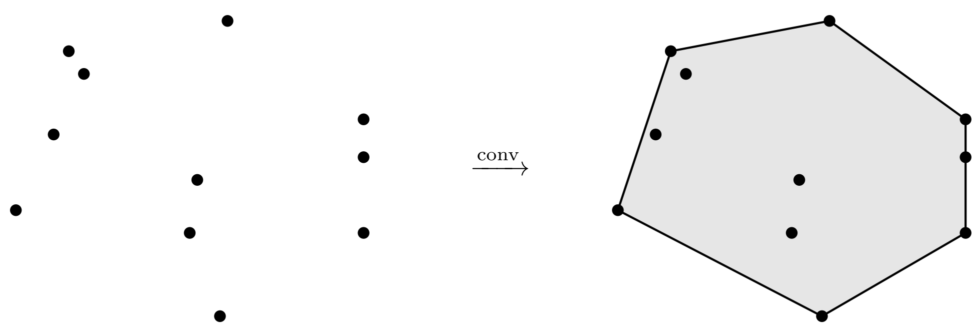
A natural question to ask is whether Pick's theorem generalizes to higher dimensions. For example, can we express the volume of a lattice polytope through some combination of $I(P)$ and $B(P)$? It turns out that higher-dimensional lattice point geometry is a lot richer than its classic two-dimensional counterpart. With this richness comes added complexity.

Definition 1.7. For points $v_1, v_2, \ldots, v_m \in \mathbb{R}^d$, we define their *convex hull* as

$$\operatorname{conv}(\{v_1, v_2, \ldots, v_m\}) := \left\{ \sum_{i=1}^{m} \lambda_i v_i : \lambda_i \geq 0,\ \sum_{i=1}^{m} \lambda_i = 1 \right\}.$$

Intuitively, think of a convex hull as stretching a rubber band around all the points and letting it snap tight.

Example 1.8. The convex hull of the left-hand points is the right-hand shaded region (with the boundary).



Definition 1.9. A region P is a *convex polytope* in $\mathbb{R}^d$ if there exists integer m and points $V = \{v_1, v_2, \ldots, v_m\}$ such that $P = \operatorname{conv}(V)$. The set of *vertices* of P is the minimal subset $V' \subseteq V$ such that $\operatorname{conv}(V') = P$. A polytope is *integral* (or a *lattice polytope*) if all of its vertices lie in $\mathbb{Z}^d$.

Definition 1.10. Points $v_0, v_1, \ldots, v_k \in \mathbb{R}^d$ are *affinely independent* if the vectors $\{v_1 - v_0, v_2 - v_0, \ldots, v_k - v_0\}$ are linearly independent. The *dimension* of a polytope P is the largest integer k such that P contains $k+1$ affinely independent points.

One can think of polygons as being built up by gluing together a bunch of triangles along their edges. This means studying just triangles yields fruit about other polygons and is thus important. Analogously, we can build up polytopes from building blocks called *simplices*.

Definition 1.11. A convex d–dimensional polytope with exactly $d+1$ vertices $v_0, v_1, \ldots, v_d$ is a *d–simplex*. The *standard d–simplex* is

$$\Delta_d = \operatorname{conv}(\{0, e_1, e_2, \ldots, e_d\}) \subset \mathbb{R}^d,$$

where e_i for $i = 1, 2, \ldots, d$ are standard basis vectors. For any simplex, its (unsigned) volume is

$$\operatorname{Vol}(\Delta) = \frac{1}{d!} \left| \det\big[v_1 - v_0 \mid v_2 - v_0 \mid \cdots \mid v_d - v_0 \big] \right|,$$

where the matrix has columns $v_i - v_0$.

For positive integer h, define the tetrahedron

$$T_h = \operatorname{conv}(\{(0,0,0),\ (1,0,0),\ (0,1,0),\ (1,1,h)\}).$$

Question 1.6 (5 pts)**.** Compute $\operatorname{Vol}(T_h)$ using the simplex volume formula.

Question 1.7 (15 pts)**.** Show that for $h \geq 1$, T_h has exactly four boundary lattice points and zero interior lattice points.

Question 1.8 (10 pts)**.** Show that no constants $\alpha, \beta \in \mathbb{R}$ can make $\operatorname{Vol}(P) = I(P) + \alpha B(P) + \beta$ hold for all integral convex 3-dimensional polytopes P.

2 Polynomials and Point–Counting

2.1 Generating Functions

A generating function is a clever way to package a whole sequence into one expression. By using the language of algebra, we can discover relationships within and between generating functions. These can then translate into identities for certain counting questions. We will exploit these ideas later to simultaneously uncover information about all the values $L_P(0), L_P(1), \ldots$ at once.

Definition 2.1. The *ordinary generating function (OGF)* of a sequence of numbers $\{a_i\}_{i=0}^{\infty} = \{a_0, a_1, \ldots\}$ is defined by

$$\sum_{k=0}^{\infty} a_k x^k.$$

Note that generating functions such as this one are formal power series, i.e., we do not consider any notion of convergence when analyzing the sums of these expressions.

Example 2.2. Consider the sequence $a_i = c^i$, where $c \neq 0$ is a constant, for all $i \geq 0$. Then, for this infinite sequence,

$$\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} c^k x^k = 1 + cx + c^2x^2 + \cdots = \frac{1}{1-cx}.$$

To see this, one can symbolically verify

$$(1-cx)(1+cx+c^2x^2+\cdots) = 1.$$

This may remind you of the infinite geometric series formula. If $a_i = c^i$ were truncated to have the first n terms, our OGF would be

$$\sum_{k=0}^{n-1} a_k x^k = \sum_{k=0}^{n-1} c^k x^k = 1 + cx + c^2x^2 + \cdots + c^{n-1}x^{n-1} = \frac{1-(cx)^n}{1-cx}$$

by a similar argument, reminiscent of the finite geometric series formula.

Question 2.1 (10 pts). Consider the famous *Frobenius Coin Problem*: given coin values $a_1, a_2, \ldots, a_d$ (where $\gcd(a_1, a_2, \ldots, a_d) = 1$), we define $p(t)$, the number of ways to make change for t units of money with as many coins of each type as we want. Find the OGF that gives the number of ways to make change for n units of money for this general case as a rational function.[a]

Question 2.2. Prove that the sequence $\left\{\binom{n+k-1}{n-1}\right\}_{k=0}^{\infty}$ is generated by the coefficients of the OGF $(1-x)^{-n}$ by:

(a) (10 pts) an algebraic argument.

(b) (10 pts) a combinatorial argument.

[a]The case of $d = 2$ is popularly referred to as the **Chicken McNugget problem** despite Chicken McNuggets originally coming in 3 differently-sized boxes.

2.2 Ehrhart's Theorem

Eugène Ehrhart (1906–2000) did not get his high school diploma until he was 22 and did not begin his PhD until his late 50s. His most impactful contribution was towards lattice point-counting under dilation on general d–polytopes, a field which we call Ehrhart Theory. One key finding in this field is that for a lattice d–polytope P, the function $L_P(t) = \#(tP \cap \mathbb{Z}^d)$ turns out to be a degree–d polynomial in t.

Theorem 2.3. (Ehrhart's Theorem) If P is an integral convex d-polytope, then $L_P(t)$ is a polynomial in $t \in \mathbb{Z}_{\geq 0}$ of degree d.

Theorem 2.4. If $P \subset \mathbb{R}^2$ is an integral convex polygon, then for $t \in \mathbb{Z}_{\geq 0}$,

$$L_P(t) = A(P)\, t^2 + \frac{B(P)}{2}\, t + 1.$$

Let $R = [0, a] \times [0, b]$ with $a, b \in \mathbb{Z}_{>0}$.

Question 2.3 (5 pts). Take the $L_P(t)$ you derived in question 1.4 and match coefficients with $A(R)$ and $B(R)$.

Theorem 2.5. For $\square_d = [0, 1]^d$, we have $L_{\square_d}(t) = (t + 1)^d$.

Question 2.4 (5 pts). Compute $I_{\square_d}(t)$ and show that for $t \geq 1$,

$$B_{\square_d}(t) = (t + 1)^d - (t - 1)^d.$$

2.3 Ehrhart Reciprocity and the h^* Polynomial

Instead of counting lattice points one dilation at a time, we can wrap these values up into a single generating function. This happens to be equivalent to a single rational function, meaning we now need only consider one mathematical object to study every value of $L_P(t)$. We will pay special attention towards the numerator of this rational function, denoted $h^*_P(z)$.

Remark 2.6. In lieu of computing $L_P(t)$ one value at a time, we consider the entire sequence $L_P(0)$, $L_P(1)$, $L_P(2)$, etc. using the generating function

$$\mathrm{Ehr}_P(z) = \sum_{t\geq 0} L_P(t)\, z^t,$$

known as the *Ehrhart series*. One can show that

$$\mathrm{Ehr}_P(z) = \frac{h_P^*(z)}{(1-z)^{d+1}}$$

is always rational, where d is the dimension of P and $h_P^*(z)$ is a polynomial with nonnegative integer coefficients.

Definition 2.7. For a permutation $\pi : \{1,\dots,d\} \to \{1,\dots,d\}$, an *ascent* is an index $i \in \{1,\dots,d-1\}$ with $\pi(i) < \pi(i+1)$. Then, $A(d,k)$ is the number of permutations on a set of size d with exactly $k-1$ ascents.

Question 2.5 (10 pts). Compute $h^*_{\square_d}(z)$ for $\square_d = [0,1]^d$ using

$$h^*_{\square_d}(z) = \sum_{k=1}^{d} A(d,k)\, z^{k-1}$$

for $d = 2$, 3, and 4.

For the following questions, write

$$\mathrm{Ehr}_P(z) = \frac{h_0 + h_1 z + \cdots + h_d z^d}{(1-z)^{d+1}}$$

Question 2.6.

(a) (15 pts) Show that for $t \in \mathbb{Z}_{\geq 0}$,

$$L_P(t) = \sum_{j=0}^{d} h_j \binom{t+d-j}{d}.$$

(b) (5 pts) Compute h_0 and h_1.

Question 2.7.

(a) (15 pts) Show that the sum of the h^*–coefficients equals the normalized volume

$$h_0 + h_1 + \cdots + h_d = d!\,\mathrm{Vol}(P).$$

(b) (5 pts) Verify part (a) for $\square_d = [0,1]^d$.

Question 2.8 (10 pts). Show that for any convex lattice polygon P, question 2.7 implies Pick's theorem.

Question 2.9. Let $p(t)$ be a polynomial of degree $\leq d$ such that

$$\sum_{t\geq 0} p(t)\, z^t \;=\; \frac{h^*(z)}{(1-z)^{d+1}}, \text{ where } h^*(z) = h_0 + h_1 z + \cdots + h_d z^d.$$

(a) (20 pts) Show that $\deg h^* \leq k$ if and only if $p(-1) = p(-2) = \cdots = p(-(d-k)) = 0$.

(b) (15 pts) If $\deg h^* = k$, prove that $p(-(d-k+1)) = (-1)^d h_k \neq 0$.

2.4 Ehrhart–Macdonald Reciprocity

As $L_P(t)$ is a polynomial, it makes sense to evaluate it at negative integers. Reciprocity says these negative values encode lattice points in the interior of P, up to a sign. Hence, the number of interior points come "for free" from the same polynomial.

Definition 2.8. For a polytope $P \subset \mathbb{R}^d$, let P° denote its interior and define

$$L_{P^\circ}(t) := \#(tP^\circ \cap \mathbb{Z}^d) \qquad (t \in \mathbb{Z}_{\geq 1}).$$

Theorem 2.9. (Ehrhart–Macdonald Reciprocity) If P is a convex lattice d-polytope, then for all $t \in \mathbb{Z}_{\geq 1}$,

$$L_P(-t) = (-1)^d \, L_{P^\circ}(t).$$

Let $T_{a,b}$ be the triangle with vertices $(0,0)$, $(a,0)$, and $(0,b)$, where $a, b \in \mathbb{Z}_{>0}$.

Question 2.10 (10 pts)**.** Let $g = \gcd(a,b)$. Show that the hypotenuse contains exactly $g+1$ lattice points, and hence

$$B(T_{a,b}) = a + b + g.$$

Question 2.11 (10 pts)**.** Use Pick's theorem to show

$$L_{T_{a,b}}(t) = \frac{ab}{2}t^2 + \frac{a+b+g}{2}t + 1.$$

Question 2.12 (10 pts)**.** Use reciprocity to express $L_{T^\circ_{a,b}}(1)$ in terms of the coefficients of $L_{T_{a,b}}(t)$.

Question 2.13 (15 pts)**.** Prove the identity

$$\gcd(a,b) = 2\sum_{k=1}^{b-1} \left\lfloor \frac{ka}{b} \right\rfloor + a + b - ab.$$

3 Counting Points Using Roots of Unity

3.1 The Sawtooth Function

In the previous section, we found that the number of lattice points in a right triangle $T_{a,b}$ depended on $\gcd(a,b)$ and a sum involving the floor function $\lfloor x \rfloor$. It turns out that point-counting arguments can help us prove other number-theoretic identities about more exotic "floor-like" functions.

Definition 3.1. The *sawtooth function* $(\!(x)\!)$ is defined as:

$$(\!(x)\!) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \in \mathbb{R} \setminus \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z} \end{cases}$$

Question 3.1 (5 pts)**.** Draw the graph of $(\!(x)\!)$. Show that it is an odd function (i.e., $(\!(-x)\!) = -(\!(x)\!)$) and periodic with period 1.

Definition 3.2. For coprime integers a and b (with $b > 0$), the *Dedekind sum* $s(a, b)$ is defined as:

$$s(a,b) = \sum_{k=1}^{b-1} \left(\left(\frac{k}{b}\right)\right)\left(\left(\frac{ka}{b}\right)\right).$$

These sums might look abstract, but they are intricately related to lattice-point counting.

Question 3.2 (10 pts). Let $P(a,b) = \sum_{k=1}^{b-1} \left(\frac{k}{b} - \frac{1}{2}\right)\left(\frac{ka}{b} - \frac{1}{2}\right)$. Using the definition of the sawtooth function, show that

$$s(a,b) = P(a,b) - \frac{1}{b}\sum_{k=1}^{b-1} k \left\lfloor \frac{ka}{b} \right\rfloor + \frac{1}{2}\sum_{k=1}^{b-1} \left\lfloor \frac{ka}{b} \right\rfloor.$$

Here, s captures geometric information. Notice that the rightmost term is exactly the number of lattice points in the right triangle $T_{b,a}$ and the second term is the related to the x-coordinate of the center of mass of those lattice points.

Question 3.3 (20 pts). Given that $\gcd(a, b) = 1$, the polynomial sum evaluates to

$$P(a,b) = \frac{a(b-1)(b-2)}{12b}$$

and the geometric interpretation of the sums $s(a, b)$, prove the reciprocity identity

$$s(a,b) + s(b,a) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a}\right).$$

The true power of this *Dedekind reciprocity law* lies in its versatility; it serves as a bridge to deep identities across disparate fields of mathematics. In Power A, the students explored its combinatorial roots via the Frobenius Coin Problem from question 2.1. Beyond these counting problems, the theory intersects with linear algebra and complex analysis to reveal even deeper structures through the language of Fourier Analysis. Ultimately, Ehrhart and Dedekind theories stretch far beyond the scope of these pages—related questions remain at the very forefront of modern research. We hope you enjoyed the journey!

3.2 A Crash Course in Fourier Analysis

The sawtooth function is a periodic function, and there is a powerful mathematical tool to study periodic functions: Fourier theory. Consider, for a positive integer b, the set of all complex-valued b-periodic functions

$$V_b = \{f : \mathbb{Z} \to \mathbb{C} \mid \forall x \in \mathbb{Z}, f(x+b) = f(x)\}.$$

For $\alpha \in \mathbb{C}$ and $f, g \in V_b$, we can define scalar multiplication by $(\alpha f)(x) = \alpha f(x)$ for all $x \in \mathbb{Z}$, and vector addition by $(f + g)(x) = f(x) + g(x)$ for all $x \in \mathbb{Z}$. It is straightforward to see that V_b is a vector space over $\mathbb{C}$ with these operations.

Question 3.4 (10 pts). Show that V_b has dimension b as a vector space over $\mathbb{C}$.

Now, we turn to investigate a different basis for this vector space.

Question 3.5 (5 pts). Consider the following function of period 3, defined by

$$\begin{aligned} n: &\quad 0,1,2,3,4,5,\cdots \\ a(n): &\quad 1,5,2,1,5,2,\cdots. \end{aligned}$$

Show that the generating function for this sequence is

$$F(z) = \sum_{n\geq 0} a(n)z^n = \frac{1+5z+2z^2}{1-z^3}.$$

Question 3.6 (10 pts). By the partial fraction expansion in question 3.5,

$$F(z) = \frac{\hat{a}(0)}{1-z} + \frac{\hat{a}(1)}{1-\omega_3 z} + \frac{\hat{a}(2)}{1-\omega_3^2 z},$$

where $\omega_3 = e^{2\pi i/3}$, a third root of unity. Find the constants $\hat{a}(0)$, $\hat{a}(1)$, and $\hat{a}(2)$.

Using the geometric series for each of these terms separately, we arrive at

$$F(z) = \sum_{n\geq 0} \left(\hat{a}(0) + \hat{a}(1)\omega_3^n + \hat{a}(2)\omega_3^{2n}\right) z^n,$$

the finite Fourier series of our sequence $a(n)$. In general, the following is true:

Theorem 3.3. Let $a(n)$ be any periodic function on $\mathbb{Z}$, with period b and let $\omega_b = e^{2\pi i/b}$. Then we can write the following finite Fourier series expansion:

$$a(n) = \sum_{k=0}^{b-1} \hat{a}(k)\omega_b^{nk},$$

where the Fourier coefficients are

$$\hat{a}(n) = \frac{1}{b}\sum_{k=0}^{b-1} a(k)\omega_b^{-nk}.$$

Question 3.7 (20 pts). Similar to what we did for Questions 3.5 and 3.6, prove Definition 3.3.

Finally, we provide a useful theorem for Fourier analysis.

Definition 3.4. Define the *inner product* on V_b as $\langle f, g\rangle = \sum_{x=0}^{b-1} f(x)\overline{g(x)}$.

Theorem 3.5. (Parseval's Identity) Let $f, g \in V_b$. Then, $\langle f, g\rangle = b\langle \hat{f}, \hat{g}\rangle$.

Question 3.8 (10 pts)**.** Show that for $k = 0, \ldots, b-1$, the set of functions $f_k(n) = \frac{1}{\sqrt{b}}\omega_b^{nk}$ forms an orthonormal basis for V_b under the inner product defined in definition 3.4.

Question 3.9 (15 pts)**.** Consider the b-periodic functions $B_b(n) = ((\frac{n}{b}))$. Show that $\hat{B}_b(n) = \frac{1}{2b}\frac{1+\omega_b^n}{1-\omega_b^n}$ for $n = 1, 2, \ldots, b-1$ and $\hat{B}_b(0) = 0$. Note that also $\hat{B}_b(n) = \frac{i}{2b}\cot\frac{\pi n}{b}$, but you do not have to show this.

Question 3.10 (10 pts)**.** Show that

$$s(a,b) = \frac{1}{4b}\sum_{m=1}^{b-1}\cot\frac{\pi m a}{b}\cot\frac{\pi m}{b}.$$

Definition 3.6. For coprime integers $a_1, \ldots, a_d, b$, and n, the Fourier-Dedekind sum is defined as

$$\sigma_n(a_1, \ldots, a_d; b) = \frac{1}{b}\sum_{k=1}^{b-1}\frac{\omega_b^{nk}}{(1-\omega_b^{ka_1})(1-\omega_b^{ka_2})\cdots(1-\omega_b^{ka_d})}.$$

Question 3.11 (10 pts)**.** We can relate the generalized Fourier-Dedekind sum back to the classical Dedekind sum. Given the identity

$$\frac{1}{a}\sum_{k=1}^{a-1}\frac{1}{1-\omega_a^k}\omega_a^{-kn} = \left\lfloor\frac{n}{a}\right\rfloor - \frac{n}{a} + \frac{1}{2} - \frac{1}{2a},$$

show that

$$\sigma_0(1, a; b) = \frac{1}{4} - \frac{1}{4b} - s(a,b).$$

Note that the formula we used somehow miraculously turned from an expression with roots of unity to a polynomial in n and a discrete quantity. We will see why in the next section.

3.3 Applications to Point–Counting

Pick's theorem tends to fail for higher dimensional polytopes. In Question 2.7 we derived one generalization, which lets us algebraically represent the volume of such a polytope. However, suppose we had a different goal: given the generalized volume of each 2-dimensional face, 3-dimensional face, etc. of the polytope, how do we figure out the number of lattice points inside the polytope? In generality, we may want to compute $L_P(t)$; Pick's theorem would solve this question for two dimensions, but the theory of Dedekind sums helps us generalize it.

Recall the *Frobenius Coin Problem* from Question 2.1: given coin values $a_1, a_2, \ldots, a_d$ (where $\gcd(a_1, a_2, \ldots, a_d) = 1$), we define $p_d(t)$, the number of ways to make change for t units of money with as many coins of each type as we want. Geometrically, this is equivalent to counting the number of non-negative integer lattice points $(x_1, \ldots, x_d)$ on the simplex defined by:

$$P_t = \{x_i \geq 0 : a_1x_1 + \cdots + a_dx_d = t\}$$

Question 3.12 (10 pts)**.** Show that $\mathrm{Vol}(P_t)$ is always a polynomial in t. Compute the degree of this polynomial.

Question 3.13 (5 pts)**.** Compute $\mathrm{Vol}(P_t)$ when $d = 2$.

If we could treat this as a continuous volume problem, the answer would simply be a polynomial in t. However, the discrete nature introduces an "error" term. We already derived the generating function $F(z) = \sum_{t=0}^{\infty} p_d(t)z^t$ for this question in Question 2.1.

To find a formula for $p_d(t)$ for general a_i, we can use a partial fraction decomposition of $F(z)$. The poles of $F(z)$ occur exactly when one of the denominators is zero, i.e., at the a_k-th roots of unity. We separate the pole at $z = 1$ (which gives the polynomial growth) from the other poles (which give the periodic behavior).

Question 3.14 (20 pts). Prove that for $d = 2$ and coprime a_1, a_2, the number of ways to make change is:[a]

$$p_2(t) = \frac{t + \frac{a_1+a_2}{2}}{a_1 a_2} + \sigma_{-t}(a_1; a_2) + \sigma_{-t}(a_2; a_1).$$

[a] Notice that the degree one term is slightly different than the continuous volume. This is because we are normalizing to a different scale–namely instead of having a 1 along the x_i axis, we have dilated to a_i, so the effective size in that direction is different. There is also a constant term, which is a correction issued by continuous boundaries of the polytope.

Finally, we return to another geometric question. The value $p_d(t)$ counts the points in the closed polytope (since $x_i \geq 0$). If we want the number of **internal** lattice points (where $x_i > 0$), we will denote this $p_d^\circ(t)$.

Question 3.15 (10 pts). Express $p_d^\circ(t)$ in terms of $p_d(t)$.

Question 3.16 (10 pts). Prove the **Ehrhart-Macdonald Reciprocity** law holds for this polytope with $d = 2$:

$$p_2^\circ(t) = p_2(-t).$$

Do not use Definition 2.9 for this question.

3.4 Higher–Dimensional Point-counting

We now have a complete picture for $d = 2$. For higher dimensions, the partial fraction decomposition takes the following form:

$$F(z) = \underbrace{\frac{C_d}{(1-z)^d} + \cdots + \frac{C_1}{(1-z)}}_{\text{Pole at } z=1} + \sum_{j=1}^{d} \sum_{k=1}^{a_j - 1} \frac{A_{j,k}}{1 - \omega_{a_j}^k z}$$

Question 3.17 (5 pts). Explain briefly why the assumption that $\gcd(a_j, a_k) = 1$ for all $j \neq k$ implies that all poles besides $z = 1$ have order 1.

The first part (the pole at $z = 1$) yields a polynomial in t (which corresponds to a "normalized" volume of the simplex). The second part (the sums over roots of unity) yields the periodic corrections. We will focus on deriving the exact form of these corrections using the Fourier-Dedekind sums.

Question 3.18 (10 pts). Fix an index $j \in \{1, \ldots, d\}$ and a specific exponent $k \in \{1, \ldots, a_j - 1\}$ and let $\lambda = \omega_{a_j}^k$. We want to find the coefficient $A_{j,k}$ in the term $\frac{A_{j,k}}{1-\lambda z}$. Using the residue formula for a simple pole, show that:

$$A_{j,k} = \frac{1}{a_j} \prod_{m \neq j} \frac{1}{1 - \omega_{a_j}^{-k a_m}}.$$

Question 3.19 (15 pts). The term derived above contributes $A_{j,k}\lambda^t$ to the coefficient of z^t in the power series expansion of $F(z)$. Show that

$$p(t) = \mathrm{Poly}_d(t) + \sum_{j=1}^{d} \sigma_{-t}(a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_d; a_j),$$

where $\mathrm{Poly}_d(t)$ is a polynomial of degree $d - 1$ (you need not derive an explicit form for this polynomial). (Hint: Consider the pole at $z = 1$.)