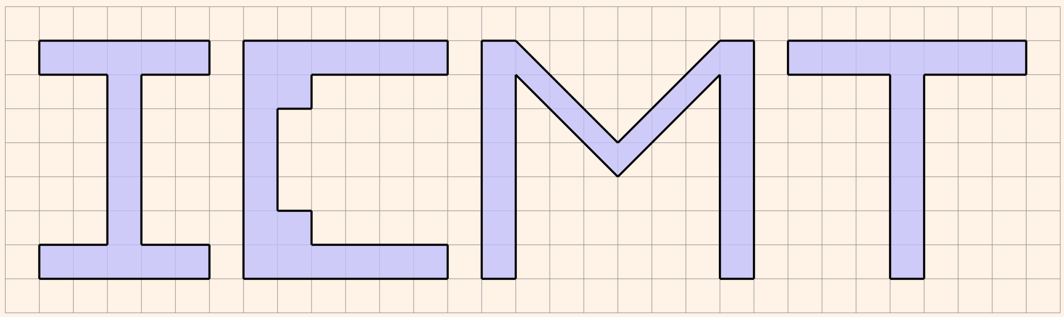
Intercollegiate Math Tournament — Power Round (Division A)
Official Solutions

1 From Polygons to Polytopes

1.1 Introducing Pick's Theorem

The grids in the following questions are partitioned into 1×1 squares and represent the 2–D lattice $\mathbb{Z}^2$.

Question 1.1 (10 pts)**.** Compute the number of interior points $I(P)$ and number of boundary points $B(P)$ for each of the four polygons. Compute the sum of the areas of these polygons via Pick's theorem.



Solution 1.1. For the "I" shape,

$$\boxed{I(P_{\text{I}}) = 0}, \qquad \boxed{B(P_{\text{I}}) = 32}, \qquad A(P_{\text{I}}) = I(P_{\text{I}}) + \frac{B(P_{\text{I}})}{2} - 1 = 0 + \frac{32}{2} - 1 = 15.$$

For the "C" shape,

$$\boxed{I(P_{\text{C}}) = 2}, \qquad \boxed{B(P_{\text{C}}) = 36}, \qquad A(P_{\text{C}}) = I(P_{\text{C}}) + \frac{B(P_{\text{C}})}{2} - 1 = 2 + \frac{36}{2} - 1 = 19.$$

For the "M" shape,

$$\boxed{I(P_{\text{M}}) = 0}, \qquad \boxed{B(P_{\text{M}}) = 42}, \qquad A(P_{\text{M}}) = I(P_{\text{M}}) + \frac{B(P_{\text{M}})}{2} - 1 = 0 + \frac{42}{2} - 1 = 20.$$

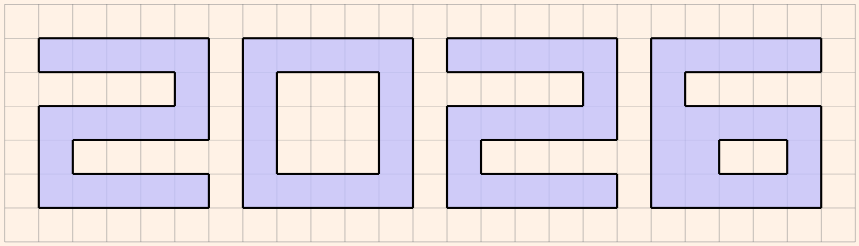
For the "T" shape,

$$\boxed{I(P_{\text{T}}) = 0}, \qquad \boxed{B(P_{\text{T}}) = 28}, \qquad A(P_{\text{T}}) = I(P_{\text{T}}) + \frac{B(P_{\text{T}})}{2} - 1 = 0 + \frac{28}{2} - 1 = 13.$$

The sum of all of these areas is

$$15 + 19 + 20 + 13 = \boxed{67}.$$

Question 1.2 (10 pts). Compute the number of interior points $I(P)$ and number of boundary points $B(P)$, including the polygons that contain holes, for each of the four polygons. Classify each polygon as simple or non-simple. Compute the sum of the areas of these polygons via Pick's theorem.



Solution 1.2. For the "2" shape, there are no holes; hence it is $\boxed{\text{simple}}$. In this case, we use Pick's theorem normally. Thus,

$$\boxed{I(P_2) = 0}, \qquad \boxed{B(P_2) = 36}, \qquad A(P_2) = I(P_2) + \frac{B(P_2)}{2} - 1 = 0 + \frac{36}{2} - 1 = 17.$$

For the "0" shape, there are holes; hence it is $\boxed{\text{non-simple}}$. Notice that both the entire "0" shape with its corresponding hole are simple shapes. We find the area of the entire region and subtract that from the area of the hole. Thus,

$$\boxed{I(P_{0,\text{total}}) = 16}, \qquad \boxed{B(P_{0,\text{total}}) = 20}, \qquad A(P_{0,\text{total}}) = I(P_{0,\text{total}}) + \frac{B(P_{0,\text{total}})}{2} - 1 = 16 + \frac{20}{2} - 1 = 25$$

and

$$\boxed{I(P_{0,\text{hole}}) = 4}, \qquad \boxed{B(P_{0,\text{hole}}) = 12}, \qquad A(P_{0,\text{hole}}) = I(P_{0,\text{hole}}) + \frac{B(P_{0,\text{hole}})}{2} - 1 = 4 + \frac{12}{2} - 1 = 9.$$

The area of the "0" shape with the hole is

$$A(P_0) = A(P_{0,\text{total}}) - A(P_{0,\text{hole}}) = 25 - 9 = 16.$$

For the "6" shape, there are holes; hence it is $\boxed{\text{non-simple}}$. Notice that both the entire "6" shape with its corresponding hole are simple shapes. We find the area of the entire region and subtract that from the area of the hole. Thus,

$$\boxed{I(P_{6,\text{total}}) = 8}, \qquad \boxed{B(P_{6,\text{total}}) = 28}, \qquad A(P_{6,\text{total}}) = I(P_{6,\text{total}}) + \frac{B(P_{6,\text{total}})}{2} - 1 = 8 + \frac{28}{2} - 1 = 21$$

and

$$\boxed{I(P_{6,\text{hole}}) = 0}, \qquad \boxed{B(P_{6,\text{hole}}) = 6}, \qquad A(P_{6,\text{hole}}) = I(P_{6,\text{hole}}) + \frac{B(P_{6,\text{hole}})}{2} - 1 = 0 + \frac{6}{2} - 1 = 2.$$

The area of the "6" shape with the hole is

$$A(P_6) = A(P_{6,\text{total}}) - A(P_{6,\text{hole}}) = 21 - 2 = 19.$$

The sum of all of these areas is

$$17 + 16 + 17 + 19 = \boxed{69}.$$

1.2 A Formal Look at Polygons and Pick's Theorem

Let $R = [0, a] \times [0, b] \subset \mathbb{R}^2$ with $a, b \in \mathbb{Z}_{>0}$.

Question 1.3 (10 pts). Compute $L(R)$, $B(R)$, and $I(R)$ in terms of a and b.

Solution 1.3. With a grid of $(a+1)\times(b+1)$ points, the number of boundary points is calculated by counting the number of points on its perimeter and subtracting this from the number of points on the corners. We get

$$\boxed{L(R) = (a+1)(b+1)}, \quad \boxed{B(R) = 2(a+b)}, \quad \boxed{I(R) = (a-1)(b-1)}.$$

Question 1.4 (10 pts). Compute $L_R(t)$ explicitly as a polynomial in $t \in \mathbb{Z}_{\geq 0}$.

Solution 1.4. We apply $L(R)$ from the previous question to the region $[0, ta] \times [0, tb]$. We get

$$\boxed{L_R(t) = ab\,t^2 + (a+b)t + 1}.$$

Question 1.5 (10 pts). Find an example of a nonconvex simple lattice polygon and verify Pick's theorem for your example.

Solution 1.5. Answers may vary. One possible example is an L–shape with coordinates $\{(0,0), (2,0), (2,1), (1,1), (1,2), (0,2)\}$. Counting the number of boundary and interior points as well as finding its area via usual techniques gives $\boxed{A = 3}$, $\boxed{B = 8}$, and $\boxed{I = 0}$. Checking with Pick's theorem, we have $A = 3 = 0 + \frac{8}{2} - 1$.

1.3 A Look at Higher Dimensional Cases

For positive integer h, define the tetrahedron

$$T_h = \operatorname{conv}(\{(0,0,0),\ (1,0,0),\ (0,1,0),\ (1,1,h)\}).$$

Question 1.6 (5 pts). Compute $\operatorname{Vol}(T_h)$ using the simplex volume formula.

Solution 1.6. By the simplex volume formula, we have

$$\det \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & h \end{pmatrix} = h \implies \boxed{\operatorname{Vol}(T_h) = \frac{h}{6}}.$$

Question 1.7 (15 pts). Show that for $h \geq 1$, T_h has exactly four boundary lattice points and zero interior lattice points.

Solution 1.7. Writing $(x, y, z) \in T_h$ as a convex combination gives

$$a(0,0,0) + b(1,0,0) + c(0,1,0) + d(1,1,h) \text{ with } a+b+c+d = 1,\ a,b,c,d \geq 0.$$

Then $x = b+d$, $y = c+d$, and $z = hd$ are our parameterized coordinates. Observe $a \geq 0$ implies $b+c+d \leq 1$. Thus, we have $x = b+d \leq 1$ and $y = c+d \leq 1$. For lattice points, each $x, y \in \{0, 1\}$. We consider the following two cases.

1. Cases $x = 0$ or $y = 0$: Then, assert $z = 0$, giving us vertices $\{(0,0,0), (1,0,0), (0,1,0)\}$.
2. Case $x = y = 1$: From $b+d = c+d = 1$ we get $b = c = 1-d$. Then $a = 1-b-c-d = 1-2(1-d)-d = d-1$. For $a \geq 0$, we need $d \geq 1$. Since $d \leq 1$ from both $d \leq 1-b \leq 1$ and $d \leq 1-c \leq 1$, we have $d = 1$, so $z = h$. Thus $(1,1,z) \in T_h \Leftrightarrow z = h$.

Therefore, $T_h \cap \mathbb{Z}^3 = \{(0,0,0), (1,0,0), (0,1,0), (1,1,h)\}$ with $B(T_h) = 4$ and $I(T_h) = 0$, as desired. □

Question 1.8 (10 pts)**.** Show that no constants $\alpha, \beta \in \mathbb{R}$ can make $\mathrm{Vol}(P) = I(P) + \alpha B(P) + \beta$ hold for all integral convex 3-polytopes P.

Solution 1.8. We argue by contradiction. Suppose $\mathrm{Vol}(P) = I(P) + \alpha B(P) + \beta$ for some constants $\alpha, \beta \in \mathbb{R}$ holds for all integral convex 3-polytopes P. Then, it must hold for a specific polytope P, say, the *Reeve tetrahedron* T_h defined for Questions 1.6 and 1.7. For this polytope T_h, we have $h/6 = 0 + 4\alpha + \beta$ for all $h \geq 1$. However, the left hand side of this equation depends on h while the right hand side is constant, a contradiction. □

2 Polynomials and Point–Counting

2.1 Generating Functions

Question 2.1 (10 pts)**.** Consider the famous *Frobenius Coin Problem*: given coin values $a_1, a_2, \ldots, a_d$ (where $\gcd(a_1, a_2, \ldots, a_d) = 1$), we define $p(t)$, the number of ways to make change for t units of money with as many coins of each type as we want. Find the OGF that gives the number of ways to make change for n units of money for this general case as a rational function.[a]

[a]The case of $d = 2$ is popularly referred to as the **Chicken McNugget problem** despite Chicken McNuggets originally coming in 3 differently-sized boxes.

Solution 2.1. Since there are infinitely many coins given for each denomination, we write down each OGF individually for each denomination. The number of ways to make change for n units of money given only a denomination of a_i units of money, $1 \leq i \leq d$ is $1 + x^{a_i} + x^{2a_i} + \cdots = \frac{1}{1-x^{a_i}}$. Therefore, the OGF that generates the number of ways to make change for n units of money with these coins is

$$\boxed{\frac{1}{(1-x^{a_1})(1-x^{a_2})\cdots(1-x^{a_{d-1}})(1-x^{a_d})} = \prod_{i=1}^{d} \frac{1}{1-x^{a_i}}}.$$

Question 2.2. Prove that the sequence $\left\{\binom{n+k-1}{n-1}\right\}_{k=0}^{\infty}$ is generated by the coefficients of the OGF $(1-x)^{-n}$ by:

(a) (10 pts) an algebraic argument.

(b) (10 pts) a combinatorial argument.

Solution 2.2. (a) **(1st Solution)** By the extended binomial theorem,

$$(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} (-x)^k (1)^{n-k} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} (-x)^k = \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} x^k. \quad \square$$

(2nd Solution) We use mathematical induction on n.

Base Case: For the case $n = 1$, we have

$$(1-x)^{-1} = \frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{k=0}^{\infty} \binom{k}{0} x^k = \sum_{k=0}^{\infty} \binom{1+k-1}{1-1} x^k.$$

Inductive Step: Suppose that $(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} x^k$. By Pascal's identity, $\binom{n+k-1}{n-1} + \binom{n+k-1}{n} = \binom{n+k}{n}$. Therefore,

$$\begin{aligned}
(1-x)^{-n} &= \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} x^k = \sum_{k=0}^{\infty} \left[\binom{n+k}{n} - \binom{n+k-1}{n} \right] x^k \\
&= \sum_{k=0}^{\infty} \binom{n+k}{n} x^k - \sum_{k=0}^{\infty} \binom{n+k-1}{n} x^k = \sum_{k=0}^{\infty} \binom{n+k}{n} x^k - \sum_{k=0}^{\infty} \binom{n+k}{n} x^{k+1} \\
&= \sum_{k=0}^{\infty} \binom{n+k}{n} x^k (1-x) = (1-x) \sum_{k=0}^{\infty} \binom{n+k}{n} x^k \\
(1-x)^{-(n+1)} &= \sum_{k=0}^{\infty} \binom{n+k}{n} x^k. \quad \square
\end{aligned}$$

(3rd Solution) Let $f(x) = (1-x)^{-n}$. Then,

$$f'(x) = (-n)(1-x)^{-(n+1)}, \qquad f''(x) = n(n+1)(1-x)^{-(n+2)}, \qquad \cdots$$

$$f^{(k)}(x) = (-1)^k n(n+1) \cdots (n+k-1)(1-x)^{-(n+k)}$$

and, by Taylor's theorem,

$$\begin{aligned}
f(x) = (1-x)^{-n} &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} (-x)^k = \sum_{k=0}^{\infty} \frac{(-1)^k n(n+1) \cdots (n+k-1)}{k!} (-x)^k = \sum_{k=0}^{\infty} \frac{(n+k-1)!}{(n-1)!k!} x^k \\
(1-x)^{-n} &= \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} x^k. \quad \square
\end{aligned}$$

(b) The binomial coefficient $\binom{n+k-1}{n-1} = \binom{n+k-1}{k}$ equals the number of ways that k can be written as an ordered sum of n nonnegative integers $a_1, a_2, \ldots, a_n$. Since a_i is a nonnegative integer for each i, we can express the choices for each a_i as $x^0 + x^1 + x^2 + \cdots = 1 + x + x^2 + \cdots = \frac{1}{1-x}$. Since there are n nonnegative integers, the resulting OGF is $\left(\frac{1}{1-x}\right)^n = (1-x)^{-n}$. $\square$

2.2 Ehrhart's Theorem

Let $R = [0, a] \times [0, b]$ with $a, b \in \mathbb{Z}_{>0}$.

Question 2.3 (5 pts)**.** Take the $L_R(t)$ you derived in question 1.4 and match coefficients with $A(R)$ and $B(R)$.

Solution 2.3. We match coefficients of $L_R(t) = ab\,t^2 + (a+b)t + 1$ (from Question 1.4) with this new definition of R.

- Coefficient of t^2: $A(R) = ab = \mathrm{Vol}_2(R)$, the volume of the polygon in two dimensions (area).
- Coefficient of t: $\frac{B(R)}{2} = a+b$, so $B(R) = 2(a+b)$, which matches the lattice point count on its perimeter.
- Constant term: $1 = \chi(R)$, the Euler characteristic for this simple polygon R.

Question 2.4 (5 pts)**.** Compute $I_{\square_d}(t)$ and show that for $t \geq 1$,

$$B_{\square_d}(t) = (t+1)^d - (t-1)^d.$$

Solution 2.4. We have $t\square_d = [0,t]^d$, so $L_{\square_d}(t) = (t+1)^d$. The coordinates of the interior points satisfy $1 \leq x_i \leq t-1$ for each $1 \leq i \leq d$, giving $I(t\square_d) = (t-1)^d$ for $t \geq 1$. Thus, $\boxed{I(t\square_d) = (t-1)^d}$ and $B(t\square_d) = (t+1)^d - (t-1)^d$, as desired. □

2.3 Ehrhart Reciprocity and the h^* Polynomial

Question 2.5 (10 pts)**.** Compute $h^*_{\square_d}(z)$ for $\square_d = [0,1]^d$ using

$$h^*_{\square_d}(z) = \sum_{k=1}^{d} A(d,k)\, z^{k-1}$$

for $d = 2$, 3, and 4.

Solution 2.5. The h^*-polynomial of the unit cube $\square_d = [0,1]^d$ is given by $h^*_{\square_d}(z) = \sum_{k=1}^{d} A(d,k)\, z^{k-1}$, where $A(d,k)$ is the Eulerian number counting permutations of $\{1, 2, \ldots, d\}$ with $k-1$ ascents.

- For $d = 2$, the Eulerian numbers are $A(2,1) = 1$, $A(2,2) = 1$, and

$$\boxed{h^*_{\square_2}(z) = 1 + z}.$$

- For $d = 3$, the Eulerian numbers are $A(3,1) = 1$, $A(3,2) = 4$, $A(3,3) = 1$, and

$$\boxed{h^*_{\square_3}(z) = 1 + 4z + z^2}.$$

- For $d = 4$, the Eulerian numbers are $A(4,1) = 1$, $A(4,2) = 11$, $A(4,3) = 11$, $A(4,4) = 1$, and

$$\boxed{h^*_{\square_4}(z) = 1 + 11z + 11z^2 + z^3}.$$

For the following questions, write

$$\mathrm{Ehr}_P(z) = \frac{h_0 + h_1 z + \cdots + h_d z^d}{(1-z)^{d+1}}$$

Question 2.6.

(a) (15 pts) Show that for $t \in \mathbb{Z}_{\geq 0}$,

$$L_P(t) = \sum_{j=0}^{d} h_j \binom{t+d-j}{d}.$$

(b) (5 pts) Compute h_0 and h_1.

Solution 2.6. (a) The $(1-z)^{d+1}$ factor in the denominator encourages us to use the result of Question 2.2. We obtain

$$\sum_{t\geq 0} L_P(t)\, z^t = \left(\sum_{j=0}^{d} h_j z^j\right)\left(\sum_{m\geq 0}\binom{m+d}{d} z^m\right) = \sum_{j=0}^{d} h_j \left(\sum_{m\geq 0}\binom{m+d}{d} z^{m+j}\right).$$

Extracting the coefficient of z^t by selecting $m = t-j$, we obtain

$$L_P(t) = \sum_{j=0}^{d} h_j \cdot \left[\text{coefficient of } z^{t-j} \text{ in } \tfrac{1}{(1-z)^{d+1}}\right] = \sum_{j=0}^{d} h_j \binom{t-j+d}{d} = \sum_{j=0}^{d} h_j \binom{t+d-j}{d},$$

as desired. □

(b)

- Setting $t = 0$, we have $L_P(0) = \sum_{j=0}^{d} h_j \binom{d-j}{d}$. Since $\binom{d-j}{d} = 0$ for $j \geq 1$, we get $L_P(0) = h_0$. However, $0P = \{(0,0,\ldots,0)\}$ has exactly one lattice point, so $\boxed{h_0 = 1}$.
- Setting $t = 1$, we have $L_P(1) = h_0\binom{d+1}{d} + h_1\binom{d}{d} + \sum_{j\geq 2} h_j \binom{d+1-j}{d}$. For $j \geq 2$, $d+1-j < d$ so $\binom{d+1-j}{d} = 0$. Thus $L_P(1) = (d+1) + h_1$, giving

$$\boxed{h_1 = L_P(1) - (d+1)}.$$

Since $L_P(1) = \#(P \cap \mathbb{Z}^d) = I(P) + B(P)$, we have $\boxed{h_1 = I(P) + B(P) - d - 1}$.

Question 2.7.

(a) (15 pts) Show that the sum of the h^*–coefficients equals the normalized volume

$$h_0 + h_1 + \cdots + h_d = d!\,\mathrm{Vol}(P).$$

(b) (5 pts) Verify part (a) for $\square_d = [0,1]^d$.

Solution 2.7. (a) From part (a) of Question 2.6, $L_P(t) = \sum_{j=0}^{d} h_j \binom{t+d-j}{d}$. As a polynomial in t, the leading term of $\binom{t+d-j}{d}$ is $\frac{t^d}{d!}$, independent of j. Therefore the leading coefficient of $L_P(t)$ is

$$[t^d]\, L_P(t) = \frac{1}{d!} \sum_{j=0}^{d} h_j.$$

By Ehrhart's theorem, the leading coefficient of $L_P(t)$ equals $\mathrm{Vol}(P)$. Hence

$$\mathrm{Vol}(P) = \frac{1}{d!} \sum_{j=0}^{d} h_j \quad \Longrightarrow \quad h_0 + h_1 + \cdots + h_d = d!\, \mathrm{Vol}(P),$$

as desired. □

(b) For $\square_d = [0,1]^d$, $\mathrm{Vol}(\square_d) = 1$, so $h_0 + h_1 + \cdots + h_d = d!$. Indeed,

- for $d = 2$, $h^*(z) = 1 + z$ with a coefficient sum of $1 + 1 = 2 = 2!$;
- for $d = 3$, $h^*(z) = 1 + 4z + z^2$ with a coefficient sum of sum $1 + 4 + 1 = 6 = 3!$; and
- for $d = 4$, $h^*(z) = 1 + 11z + 11z^2 + z^3$ with a coefficient sum of sum $1 + 11 + 11 + 1 = 24 = 4!$.

You can check also with the identity $\sum_{k=1}^{d} A(d,k) = d!$.

Question 2.8 (10 pts). Show that for any convex lattice polygon P, question 2.7 implies Pick's theorem.

Solution 2.8. Let P be a convex lattice polygon (so $d = 2$). From Question 2.6(a),

$$\begin{aligned} L_P(t) &= h_0 \binom{t+2}{2} + h_1 \binom{t+1}{2} + h_2 \binom{t}{2} \\ &= h_0 \frac{(t+2)(t+1)}{2} + h_1 \frac{t(t+1)}{2} + h_2 \frac{t(t-1)}{2} \\ L_P(t) &= \frac{h_0 + h_1 + h_2}{2}\, t^2 + \frac{3h_0 + h_1 - h_2}{2}\, t + h_0 \end{aligned}$$

By Ehrhart's theorem for $d = 2$, $L_P(t) = A(P)\, t^2 + \frac{B(P)}{2}\, t + 1$. By matching the leading coefficients, we have

$$A(P) = \frac{h_0 + h_1 + h_2}{2} \implies h_0 + h_1 + h_2 = 2A(P) = 2!A(P),$$

which matches the desired form from Question 2.7(a). Evaluating $L_P(t)$ at $t = 1$, we have

$$L_P(1) = A(P) + \frac{B(P)}{2} + 1 = I(P) + B(P),$$

where $L_P(1) = I(P) + B(P)$ counts all lattice points in P. Rearranging this equation, we get

$$A(P) = I(P) + \frac{B(P)}{2} - 1,$$

which is Pick's theorem. □

Question 2.9. Let $p(t)$ be a polynomial of degree $\le d$ such that

$$\sum_{t\ge 0} p(t)\, z^t \;=\; \frac{h^*(z)}{(1-z)^{d+1}}, \text{ where } h^*(z) = h_0 + h_1 z + \cdots + h_d z^d.$$

(a) (20 pts) Show that $\deg h^* \le k$ if and only if $p(-1) = p(-2) = \cdots = p(-(d-k)) = 0$.

(b) (15 pts) If $\deg h^* = k$, prove that $p(-(d-k+1)) = (-1)^d h_k \neq 0$.

Solution 2.9. (a) We use the formula $L_P(t) = p(t) = \sum_{j=0}^{d} h_j \binom{t+d-j}{d}$ from Question 2.6(a). For negative integers $t = -m$ with $m \ge 1$,

$$p(-m) = \sum_{j=0}^{d} h_j \binom{d-j-m}{d}.$$

Now $\binom{d-j-m}{d} = 0$ whenever $0 \le d-j-m < d$, i.e., when $1 \le j+m \le d$. For $j+m > d$, we use the identity $\binom{n}{d} = (-1)^d \binom{d-1-n}{d}$ for $n < 0$ (extended binomial theorem), giving $\binom{d-j-m}{d} = (-1)^d \binom{j+m-1}{d}$. Therefore, for $1 \le m \le d$:

$$p(-m) = (-1)^d \sum_{j=d-m+1}^{d} h_j \binom{j+m-1}{d}. \tag{1}$$

($\Rightarrow$) If $\deg h^* \le k$, then $h_j = 0$ for all $j > k$. For $m \in \{1, 2, \ldots, d-k\}$, the sum in (1) runs over $j > d-m \ge d-(d-k) = k$, so every h_j in the sum vanishes. Hence $p(-m) = 0$ for each $m \in \{1, 2, \ldots, d-k\}$.

($\Leftarrow$) Suppose $p(-1) = p(-2) = \cdots = p(-(d-k)) = 0$. We prove $h_d = h_{d-1} = \cdots = h_{k+1} = 0$ by downward induction on m.

- *Base Case*:
  - For the case $m = 1$, $p(-1) = (-1)^d h_d \binom{d}{d} = (-1)^d h_d = 0$, so $h_d = 0$.
  - For the case $m = 2$, $p(-2) = (-1)^d \left[h_{d-1}\binom{d}{d} + h_d \binom{d+1}{d}\right] = (-1)^d h_{d-1} = 0$ (using $h_d = 0$), so $h_{d-1} = 0$.
- *Inductive Step*: At step $m = \ell$, $h_d = h_{d-1} = \cdots = h_{d-\ell+2} = 0$, so $p(-\ell) = (-1)^d h_{d-\ell+1} \cdot 1 = 0$, giving $h_{d-\ell+1} = 0$. At $m = d-k$, we obtain $h_{k+1} = 0$.

Thus $h_{k+1} = h_{k+2} = \cdots = h_d = 0$, i.e. $\deg h^* \le k$. □

(b) If $\deg h^* = k$, then $h_k \neq 0$ and $h_{k+1} = h_{k+2} = \cdots = h_d = 0$. Setting $m = d-k+1$ in (1):

$$p(-(d-k+1)) = (-1)^d \sum_{j=k}^{d} h_j \binom{j+d-k}{d}.$$

Since $h_{k+1} = h_{k+2} = \cdots = h_d = 0$, only the $j = k$ term survives, leaving us with

$$p(-(d-k+1)) = (-1)^d h_k \binom{k+d-k}{d} = (-1)^d h_k \binom{d}{d} = (-1)^d h_k.$$

Since $\deg h^* = k$, we have $h_k \neq 0$, so

$$p(-(d-k+1)) = (-1)^d\, h_k \binom{d}{d} = (-1)^d h_k \neq 0,$$

as desired. □

2.4 Ehrhart–Macdonald Reciprocity

Let $T_{a,b}$ be the triangle with vertices $(0,0)$, $(a,0)$, and $(0,b)$, where $a, b \in \mathbb{Z}_{>0}$.

Question 2.10 (10 pts). Let $g = \gcd(a,b)$. Show that the hypotenuse contains exactly $g+1$ lattice points, and hence

$$B(T_{a,b}) = a + b + g.$$

Solution 2.10. Let $g = \gcd(a,b)$. The hypotenuse is the segment from $(a,0)$ to $(0,b)$, whose direction vector is $\langle -a, b\rangle$. For this line segment, it contains $\gcd(a,b)+1 = g+1$ lattice points. The other two sides contribute $(a+1)$ points on the x-axis and $(b+1)$ points on the y-axis. Adding these quantities and subtracting the 3 corner vertices counted twice gives

$$B(T_{a,b}) = (a+1) + (b+1) + (g+1) - 3 = a + b + g,$$

as desired. □

Question 2.11 (10 pts). Use Pick's theorem to show

$$L_{T_{a,b}}(t) = \frac{ab}{2}t^2 + \frac{a+b+g}{2}t + 1.$$

Solution 2.11. The area of this triangle is $A(T_{a,b}) = \frac{ab}{2}$ and, from Question 2.10, $B(T_{a,b}) = a+b+g$. Therefore, by Ehrhart's theorem for $d = 2$, we have

$$L_{T_{a,b}}(t) = \frac{ab}{2}\,t^2 + \frac{a+b+g}{2}\,t + 1.$$

as desired. □

Question 2.12 (10 pts). Use reciprocity to express $L_{T^\circ_{a,b}}(1)$ in terms of the coefficients of $L_{T_{a,b}}(t)$.

Solution 2.12. For $d = 2$, reciprocity says $L_P(-t) = L_{P^\circ}(t)$, so

$$L_{T^\circ_{a,b}}(1) = L_{T_{a,b}}(-1) = \frac{ab}{2} - \frac{a+b+g}{2} + 1 = \boxed{\frac{ab-a-b-g+2}{2}}.$$

Question 2.13 (15 pts). Prove the identity

$$\gcd(a,b) = 2\sum_{k=1}^{b-1}\left\lfloor \frac{ka}{b} \right\rfloor + a + b - ab.$$

Solution 2.13. We count the total number of interior lattice points of $T_{a,b}$ by counting up the number of interior points from each horizontal line $y = k$ for $k = 1, 2, \ldots, b-1$. The hypotenuse of $T_{a,b}$ is defined by the equation $\frac{x}{a} + \frac{y}{b} = 1$, i.e. $x = a\left(1 - \frac{k}{b}\right)$. The x-coordinates of the interior points of $T_{a,b}$ satisfy $0 < x < a\left(1 - \frac{k}{b}\right)$, so at a particular height $y = k$, the number of interior points is

$$\#\left\{x \in \mathbb{Z} : 1 \le x < a\left(1 - \frac{k}{b}\right)\right\} = \left\lfloor \frac{a(b-k)-1}{b} \right\rfloor = a - 1 - \left\lfloor \frac{ak}{b} \right\rfloor.$$

Summing over $k = 1, 2, \ldots, b-1$ gives

$$L_{T^\circ_{a,b}}(1) = \sum_{k=1}^{b-1}\left(a - 1 - \left\lfloor \frac{ak}{b} \right\rfloor\right) = (a-1)(b-1) - \sum_{k=1}^{b-1}\left\lfloor \frac{ak}{b} \right\rfloor.$$

Since $L_{T^\circ_{a,b}}(1) = \frac{ab-a-b-g+2}{2}$ from Question 2.12, solving for $g = \gcd(a,b)$ gives

$$2\sum_{k=1}^{b-1}\left\lfloor \frac{ak}{b} \right\rfloor = ab - a - b + g \quad \Longleftrightarrow \quad g = 2\sum_{k=1}^{b-1}\left\lfloor \frac{ak}{b} \right\rfloor + a + b - ab,$$

as desired. □

3 Counting Points Using Roots of Unity

3.1 The Sawtooth Function

Question 3.1 (5 pts)**.** Draw the graph of $((x))$. Show that it is an odd function (i.e., $((-x)) = -((x))$) and periodic with period 1.

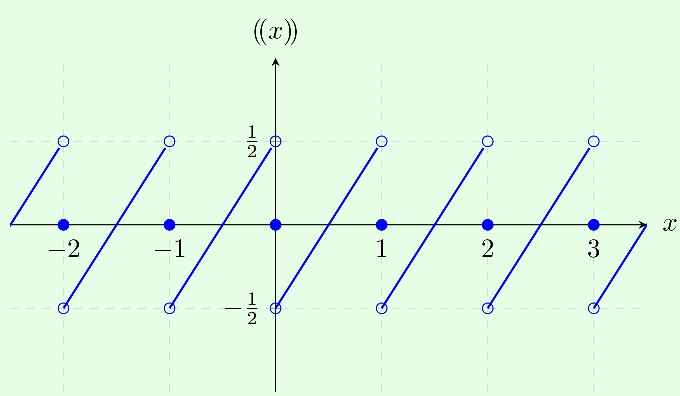
Solution 3.1. First, we show periodicity. For any $x \in \mathbb{R} \setminus \mathbb{Z}$, we see that

$$((x+1)) = (x+1) - \lfloor x+1 \rfloor - \frac{1}{2} = (x+1) - (\lfloor x \rfloor + 1) - \frac{1}{2} = x - \lfloor x \rfloor - \frac{1}{2} = ((x)).$$

For $x \in \mathbb{Z}$, $((x+1)) = 0 = ((x))$. Thus, it is periodic with period 1. Next, we show it is an odd function. For $x \in \mathbb{Z}$, $((-x)) = 0 = -0 = -((x))$. For $x \in \mathbb{R} \setminus \mathbb{Z}$, we use the property that $\lfloor -x \rfloor = -\lfloor x \rfloor - 1$ to see that

$$((-x)) = -x - \lfloor -x \rfloor - \frac{1}{2} = -x - (-\lfloor x \rfloor - 1) - \frac{1}{2} = -x + \lfloor x \rfloor + \frac{1}{2} = -\left(x - \lfloor x \rfloor - \frac{1}{2}\right) = -((x)). \quad \square$$

The graph of $((x))$ is shown below.



Question 3.2 (10 pts)**.** Let $P(a,b) = \sum_{k=1}^{b-1}\left(\frac{k}{b}-\frac{1}{2}\right)\left(\frac{ka}{b}-\frac{1}{2}\right)$. Using the definition of the sawtooth function, show that

$$s(a,b) = P(a,b) - \frac{1}{b}\sum_{k=1}^{b-1} k\left\lfloor\frac{ka}{b}\right\rfloor + \frac{1}{2}\sum_{k=1}^{b-1}\left\lfloor\frac{ka}{b}\right\rfloor.$$

Solution 3.2. By definition, $s(a,b) = \sum_{k=1}^{b-1}\left(\left(\frac{k}{b}\right)\right)\left(\left(\frac{ka}{b}\right)\right)$. Because $0 < 1 \le k \le b-1 < b$, we have $0 < \frac{k}{b} < 1$, which means $\lfloor\frac{k}{b}\rfloor = 0$. Thus, $\left(\left(\frac{k}{b}\right)\right) = \frac{k}{b} - \frac{1}{2}$. We see that

$$\begin{aligned}
s(a,b) &= \sum_{k=1}^{b-1}\left(\frac{k}{b}-\frac{1}{2}\right)\left(\frac{ka}{b}-\left\lfloor\frac{ka}{b}\right\rfloor-\frac{1}{2}\right)\\
&= \sum_{k=1}^{b-1}\left(\frac{k}{b}-\frac{1}{2}\right)\left(\frac{ka}{b}-\frac{1}{2}\right) - \sum_{k=1}^{b-1}\left(\frac{k}{b}-\frac{1}{2}\right)\left\lfloor\frac{ka}{b}\right\rfloor\\
s(a,b) &= P(a,b) - \sum_{k=1}^{b-1}\frac{k}{b}\left\lfloor\frac{ka}{b}\right\rfloor + \frac{1}{2}\sum_{k=1}^{b-1}\left\lfloor\frac{ka}{b}\right\rfloor,
\end{aligned}$$

as desired. □

Question 3.3 (20 pts)**.** Given that the polynomial sum evaluates to $P(a,b) = \frac{a(b-1)(b-2)}{12b}$ and the geometric interpretation of the sums $s(a,b)$, prove the reciprocity identity

$$s(a,b) + s(b,a) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a}\right).$$

Solution 3.3. We use the expression for $s(a,b)$ derived in Question 3.2 to obtain

$$\begin{aligned}
s(a,b)+s(b,a) &= P(a,b) - \frac{1}{b}\sum_{k=1}^{b-1}k\left\lfloor\frac{ka}{b}\right\rfloor + \frac{1}{2}\sum_{k=1}^{b-1}\left\lfloor\frac{ka}{b}\right\rfloor + P(b,a) - \frac{1}{a}\sum_{k=1}^{a-1}k\left\lfloor\frac{kb}{a}\right\rfloor + \frac{1}{2}\sum_{k=1}^{a-1}\left\lfloor\frac{kb}{a}\right\rfloor\\
&= \left[P(a,b)+P(b,a)\right] - \left[\frac{1}{b}\sum_{k=1}^{b-1}k\left\lfloor\frac{ka}{b}\right\rfloor + \frac{1}{a}\sum_{k=1}^{a-1}k\left\lfloor\frac{kb}{a}\right\rfloor\right] + \left[\frac{1}{2}\sum_{k=1}^{b-1}\left\lfloor\frac{ka}{b}\right\rfloor + \frac{1}{2}\sum_{k=1}^{a-1}\left\lfloor\frac{kb}{a}\right\rfloor\right]\\
s(a,b)+s(b,a) &= I_1 - I_2 + I_3.
\end{aligned}$$

Using the given expression for $P(a,b)$ gives

$$I_1 = P(a,b) + P(b,a) = \frac{a(b-1)(b-2)}{12b} + \frac{b(a-1)(a-2)}{12a}.$$

Using Question 2.13 with $g = \gcd(a,b) = 1$ on the rightmost term of $s(a,b)+s(b,a)$ gives

$$I_3 = \frac{1}{2}\sum_{k=1}^{b-1}\left\lfloor\frac{ka}{b}\right\rfloor + \frac{1}{2}\sum_{k=1}^{a-1}\left\lfloor\frac{kb}{a}\right\rfloor = \frac{(a-1)(b-1)}{2}.$$

Solution 3.3. (cont.) We now focus on I_2, the middle term of $s(a,b)+s(b,a)$, for the rest of the required computations. For this term, we cannot consider both sections individually. For arbitrary a, b such that $\gcd(a,b)=1$, we want to geometrically evaluate the sum

$$aS_1 + bS_2, \text{ where } S_1 = \sum_{k=1}^{b-1} k\left\lfloor \frac{ka}{b}\right\rfloor \text{ and } S_2 = \sum_{k=1}^{a-1} k\left\lfloor \frac{kb}{a}\right\rfloor.$$

Let R be the set of interior integer lattice points in the rectangle $(0,b)\times(0,a)$. The diagonal line $y=\frac{a}{b}x$ (or $ax=by$) divides R into a lower triangle T_1 and an upper triangle T_2. Because $\gcd(a,b)=1$, no integer points lie exactly on the diagonal. Notice that the number of lattice points in the k-th column of T_1 is exactly $\lfloor \frac{ka}{b}\rfloor$. Thus, multiplying by the column index k and summing over all columns gives the sum of the x-coordinates of all points in T_1. Therefore, $S_1 = \sum_{T_1} x$. By identical logic for the rows of T_2, $S_2 = \sum_{T_2} y$. We can now rewrite our target sum over the points in the regions as

$$aS_1 + bS_2 = \sum_{(x,y)\in T_1} ax + \sum_{(x,y)\in T_2} by.$$

For any point in T_1, $ax > by$, meaning $\max(ax,by)=ax$. For any point in T_2, $ax<by$, meaning $\max(ax,by)=by$. Since T_1 and T_2 partition the rectangle R, we can unify the sum as

$$aS_1 + bS_2 = \sum_{(x,y)\in R} \max(ax,by).$$

Using the algebraic identity $\max(A,B)=\frac{A+B+|A-B|}{2}$, we split the sum over R by

$$\sum_R \max(ax,by) = \frac{1}{2}\sum_R (ax+by) + \frac{1}{2}\sum_R |ax-by|.$$

The first term is a straightforward arithmetic sum over the independent variables x and y.

$$\frac{1}{2}\sum_R (ax+by) = \frac{1}{2}\left(a\sum_{x=1}^{b-1}\sum_{y=1}^{a-1} x + b\sum_{x=1}^{b-1}\sum_{y=1}^{a-1} y\right) = \frac{1}{2}\left(a(a-1)\frac{b(b-1)}{2} + b(b-1)\frac{a(a-1)}{2}\right) = \frac{ab(a-1)(b-1)}{2}$$

For the second term, observe that the map $(x,y)\mapsto(b-x,a-y)$ is a bijection on R that negates the value of $ax-by$. Therefore, the absolute values are perfectly symmetric, and $\frac{1}{2}\sum_R |ax-by| = \sum_{T_1}(ax-by)$. We evaluate this sum column by column over T_1. Then,

$$\sum_{T_1}(ax-by) = \sum_{x=1}^{b-1}\sum_{y=1}^{\lfloor \frac{ax}{b}\rfloor}(ax-by) = \sum_{x=1}^{b-1}\left(ax\left\lfloor\frac{ax}{b}\right\rfloor - \frac{b}{2}\left\lfloor\frac{ax}{b}\right\rfloor\left(\left\lfloor\frac{ax}{b}\right\rfloor+1\right)\right)$$

We substitute the fractional part $\{\frac{ax}{b}\} = \frac{ax}{b} - \lfloor\frac{ax}{b}\rfloor$ to remove the floors. Remarkably, when expanding the squared term, the cross-terms perfectly cancel out the $ax\{\frac{ax}{b}\}$ terms, leaving

$$\sum_{T_1}(ax-by) = \sum_{x=1}^{b-1}\left(\frac{a^2x^2}{2b} - \frac{b}{2}\left\{\frac{ax}{b}\right\}^2 - \frac{ax}{2} + \frac{b}{2}\left\{\frac{ax}{b}\right\}\right).$$

Because $\gcd(a,b)=1$, the fractional parts $\{\frac{ax}{b}\}$ exactly permute the values $\frac{1}{b},\frac{2}{b},\dots,\frac{b-1}{b}$ as x ranges from 1 to $b-1$. Substituting standard formulas for the sums of integers and squares yields

$$\sum_{T_1}(ax-by) = \frac{a^2(b-1)(2b-1)}{12} - \frac{(b-1)(2b-1)}{12} - \frac{ab(b-1)}{4} + \frac{b(b-1)}{4} = \frac{(a-1)(b-1)(2ab-a-b-1)}{12}$$

Solution 3.3. (cont.) Plugging the closed forms solutions for these two terms yields

$$aS_1 + bS_2 = \sum_R \max(ax, by) = \frac{1}{2}\sum_R (ax+by) + \frac{1}{2}\sum_R |ax-by|$$
$$= \frac{ab(a-1)(b-1)}{2} + \frac{(a-1)(b-1)(2ab-a-b-1)}{12}.$$

The remaining sum needed is $I_2 = \frac{1}{b}S_1 + \frac{1}{a}S_2 = \frac{1}{ab}(aS_1 + bS_2)$. Thus,

$$\begin{aligned}
I_2 &= \frac{1}{ab}\left(\frac{ab(a-1)(b-1)}{2} + \frac{(a-1)(b-1)(2ab-a-b-1)}{12}\right)\\
&= \frac{(a-1)(b-1)}{2} + \frac{(a-1)(b-1)}{12}\left(2 - \frac{1}{b} - \frac{1}{a} - \frac{1}{ab}\right)\\
&= \frac{2(a-1)(b-1)}{3} - \frac{(a-1)(b-1)}{12}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{ab}\right)\\
I_2 &= \frac{2(a-1)(b-1)}{3} - \frac{(a-1)(b-1)}{12}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{ab}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
-I_2 + I_3 &= -\left[\frac{2(a-1)(b-1)}{3} - \frac{(a-1)(b-1)}{12}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{ab}\right)\right] + \frac{(a-1)(b-1)}{2}\\
&= -\frac{(a-1)(b-1)}{6} + \frac{(a-1)(b-1)}{12}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{ab}\right)
\end{aligned}$$

and

$$\begin{aligned}
s(a,b) + s(b,a) &= I_1 + (-I_2 + I_3)\\
&= \left[\frac{a(b-1)(b-2)}{12b} + \frac{b(a-1)(a-2)}{12a}\right] + \left[-\frac{(a-1)(b-1)}{6} + \frac{(a-1)(b-1)}{12}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{ab}\right)\right]\\
&= \frac{a(b^2-3b+2)}{12b} + \frac{b(a^2-3a+2)}{12a} - \frac{ab-a-b+1}{6} + \frac{(ab-a-b+1)(a+b+1)}{12ab}\\
&= \frac{ab}{12} - \frac{a}{4} + \frac{a}{6b} + \frac{ab}{12} - \frac{b}{4} + \frac{b}{6a} - \frac{ab}{6} + \frac{a}{6} + \frac{b}{6} - \frac{1}{6} + \frac{\left(1 - \frac{1}{a} - \frac{1}{b} + \frac{1}{ab}\right)(a+b+1)}{12}\\
&= \frac{ab}{6} - \frac{a+b}{4} + \frac{a}{6b} + \frac{b}{6a} - \frac{ab}{6} + \frac{a+b}{6} - \frac{1}{6} + \frac{a+b+1-\frac{a}{b}-1-\frac{1}{b}-1-\frac{b}{a}-\frac{1}{a}+\frac{1}{b}+\frac{1}{a}+\frac{1}{ab}}{12}\\
&= -\frac{1}{6} - \frac{a+b}{12} + \frac{a}{6b} + \frac{b}{6a} + \frac{a+b}{12} - \frac{a}{12b} - \frac{b}{12a} + \frac{1}{12ab} - \frac{1}{12}\\
&= -\frac{1}{4} + \frac{a}{12b} + \frac{b}{12a} + \frac{1}{12ab}\\
s(a,b) + s(b,a) &= -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a}\right),
\end{aligned}$$

as desired. □

3.2 A Crash Course in Fourier Analysis

Question 3.4 (10 pts). Show that V_b has dimension b as a vector space over $\mathbb{C}$.

Solution 3.4. A function $f \in V_b$ is completely determined by its values on a single period, for example, at the integers $x \in \{0, 1, \ldots, b-1\}$. We can construct a natural basis consisting of b indicator functions. Let $\delta_k \in V_b$ be defined for $k \in \{0, 1, \ldots, b-1\}$ such that:

$$\delta_k(x) = \begin{cases} 1 & \text{if } x \equiv k \pmod b \\ 0 & \text{otherwise} \end{cases}.$$

Any function $f \in V_b$ can be uniquely expressed as a linear combination of these basis functions: $f = \sum_{k=0}^{b-1} f(k)\delta_k$. Furthermore, such basis functions are clearly linearly independent. Suppose there exists $c_0, c_1, \ldots, c_{b-1}$ such that $\sum_{k=0}^{b-1} c_k\delta_k = 0$. Then, for any i, we have that $0 = \sum_{k=0}^{b-1} c_k\delta_k(i) = c_i \cdot 1 + 0 = c_i$. This implies all the c_i are 0, showing linear independence. Since there are exactly b such basis functions and they are linearly independent, the dimension of the vector space V_b is b. □

Question 3.5 (5 pts). Consider the following function of period 3, defined by

$$\begin{aligned} n: &\quad 0, 1, 2, 3, 4, 5, \cdots \\ a(n): &\quad 1, 5, 2, 1, 5, 2, \cdots. \end{aligned}$$

Show that the generating function for this sequence is

$$F(z) = \sum_{n \geq 0} a(n)z^n = \frac{1 + 5z + 2z^2}{1 - z^3}.$$

Solution 3.5. The sequence $1, 5, 2, 1, 5, 2, \ldots$ has a generating function of

$$F(z) = 1 + 5z + 2z^2 + z^3 + 5z^4 + 2z^5 + z^6 + \cdots.$$

We can factor out a factor of $z^{3(k-1)}$, $k = 1, 2, \ldots$, for every k-th block of three terms, as follows.

$$F(z) = (1 + 5z + 2z^2) + z^3(1 + 5z + 2z^2) + z^6(1 + 5z + 2z^2) + \ldots$$

$$F(z) = (1 + 5z + 2z^2)(1 + z^3 + z^6 + \cdots)$$

Recognizing the infinite geometric series $1 + z^3 + z^6 + \cdots = \frac{1}{1-z^3}$, we get the final form

$$F(z) = \frac{1 + 5z + 2z^2}{1 - z^3}$$

as desired. □

Question 3.6 (10 pts). By the partial fraction expansion in the previous question,

$$F(z) = \frac{\hat{a}(0)}{1 - z} + \frac{\hat{a}(1)}{1 - \omega_3 z} + \frac{\hat{a}(2)}{1 - \omega_3^2 z},$$

where $\omega_3 = e^{2\pi i/3}$, a third root of unity. Find the constants $\hat{a}(0)$, $\hat{a}(1)$, and $\hat{a}(2)$. Express your answer in the form $a + bi$ for real numbers a, b.

Solution 3.6. We are given $F(z) = \frac{1+5z+2z^2}{1-z^3} = \frac{\hat{a}(0)}{1-z} + \frac{\hat{a}(1)}{1-\omega_3 z} + \frac{\hat{a}(2)}{1-\omega_3^2 z}$. To find each constant, we can multiply $F(z)$ by the corresponding denominator and evaluate the limit as z approaches the pole. Note that $1 - z^3 = (1-z)(1-\omega_3 z)(1-\omega_3^2 z)$. For $\hat{a}(0)$, we multiply by $(1-z)$ and take $z \to 1$ to obtain

$$\hat{a}(0) = \lim_{z\to 1} \frac{1+5z+2z^2}{1+z+z^2} = \frac{1+5(1)+2(1)}{1+1+1} = \boxed{\frac{8}{3}}.$$

For $\hat{a}(1)$, we multiply by $(1-\omega_3 z)$ and evaluate at $z = \omega_3^{-1} = \omega_3^2$ to obtain

$$\hat{a}(1) = \left.\frac{1+5z+2z^2}{(1-z)(1-\omega_3^2 z)}\right|_{z=\omega_3^2} = \frac{1+5\omega_3^2+2\omega_3^4}{(1-\omega_3^2)(1-\omega_3)} = \frac{1+2\omega_3+5\omega_3^2}{3} = \boxed{-\frac{5}{6} - i\frac{\sqrt{3}}{2}}.$$

For $\hat{a}(2)$, we multiply by $(1-\omega_3^2 z)$ and evaluate at $z = \omega_3^{-2} = \omega_3$ to obtain

$$\hat{a}(2) = \left.\frac{1+5z+2z^2}{(1-z)(1-\omega_3 z)}\right|_{z=\omega_3} = \frac{1+5\omega_3+2\omega_3^2}{(1-\omega_3)(1-\omega_3^2)} = \frac{1+5\omega_3+2\omega_3^2}{3} = \boxed{-\frac{5}{6} + i\frac{\sqrt{3}}{2}}.$$

Question 3.7 (20 pts). Similar to what we did for the previous two questions, prove Theorem 3.3.

Solution 3.7. We wish to show that $a(n) = \sum_{m=0}^{b-1} \hat{a}(m)\omega_b^{nm}$. We substitute the provided formula for the Fourier coefficients, $\hat{a}(m) = \frac{1}{b}\sum_{k=0}^{b-1} a(k)\omega_b^{-mk}$, to obtain

$$a(n) = \sum_{m=0}^{b-1}\left(\frac{1}{b}\sum_{k=0}^{b-1} a(k)\omega_b^{-mk}\right)\omega_b^{nm}$$

Since these are finite sums, we can rearrange the order of summation to obtain

$$a(n) = \frac{1}{b}\sum_{k=0}^{b-1} a(k) \sum_{m=0}^{b-1} \omega_b^{m(n-k)}.$$

We now evaluate the inner sum over m.

- Case 1: If $n \equiv k \pmod{b}$, then $\omega_b^{n-k} = 1$. The inner sum becomes $\sum_{m=0}^{b-1} 1 = b$ and the expression on the right-hand side evaluates to $\frac{1}{b}a(n)\cdot b = a(n)$.
- Case 2: If $n \not\equiv k \pmod{b}$, then $\omega_b^{n-k} \neq 1$. The inner sum is a finite geometric series that evaluates to

  $$\sum_{m=0}^{b-1}(\omega_b^{n-k})^m = \frac{1-(\omega_b^{n-k})^b}{1-\omega_b^{n-k}} = \frac{1-1}{1-\omega_b^{n-k}} = 0.$$

  Because the inner sum is zero for all $k \not\equiv n \pmod{b}$, all terms vanish except for the single term where $k = n \pmod{b}$. For that term, the expression evaluates to $\frac{1}{b}a(n)\cdot b = a(n)$.

This completes the proof. □

Question 3.8 (10 pts)**.** Show that for $k = 0, 1, \ldots, b-1$, the set of functions $f_k(n) = \frac{1}{\sqrt{b}}\omega_b^{nk}$ forms an orthonormal basis for V_b under the inner product defined in Definition 3.4.

Solution 3.8. We check the inner product of f_j and f_k. By Definition 3.4,

$$\langle f_j, f_k\rangle = \sum_{n=0}^{b-1} f_j(n)\overline{f_k(n)} = \sum_{n=0}^{b-1}\left(\frac{1}{\sqrt{b}}\omega_b^{nj}\right)\left(\frac{1}{\sqrt{b}}\omega_b^{-nk}\right) = \frac{1}{b}\sum_{n=0}^{b-1}\omega_b^{n(j-k)}.$$

We consider two cases.

- Case 1: If $j = k$, then $\omega_b^{n(j-k)} = \omega_b^0 = 1$. The sum is $\frac{1}{b}\sum_{n=0}^{b-1} 1 = \frac{b}{b} = 1$.
- Case 2: If $j \neq k$, let $\lambda = \omega_b^{j-k}$. Since $0 \le j, k \le b-1$ and $j \neq k$, $\lambda \neq 1$. The sum is a finite geometric series that evaluates to

$$\frac{1}{b}\sum_{n=0}^{b-1}\lambda^n = \frac{1}{b}\frac{1-\lambda^b}{1-\lambda}$$

  Since $\lambda^b = (\omega_b^b)^{j-k} = 1^{j-k} = 1$, the numerator is zero, so the inner product is 0.

Thus, the set is orthonormal. Since there are b such vectors and the dimension of V_b is b, they form an orthonormal basis. $\square$

Question 3.9 (15 pts)**.** Consider the b-periodic functions $B_b(n) = ((\frac{n}{b}))$. Show that $\hat{B}_b(n) = \frac{1}{2b}\frac{1+\omega_b^n}{1-\omega_b^n}$ for $n = 1, 2, \ldots, b-1$ and $\hat{B}_b(0) = 0$. Note that also $\hat{B}_b(n) = \frac{i}{2b}\cot\frac{\pi n}{b}$, but you do not have to show this.

Solution 3.9. We have that

$$\hat{B}_b(n) = \frac{1}{b}\sum_{k=0}^{b-1} B_b(k)\omega_b^{-nk} = \frac{1}{b}\sum_{k=0}^{b-1}\left(\left(\frac{k}{b}\right)\right)\omega_b^{-nk} = \frac{1}{b}\left(0 + \sum_{k=1}^{b-1}\left(\frac{k}{b} - \left\lfloor\frac{k}{b}\right\rfloor - \frac{1}{2}\right)\omega_b^{-nk}\right)$$

$$\hat{B}_b(n) = \frac{1}{b}\left(\sum_{k=1}^{b-1}\left(\frac{k}{b} - \frac{1}{2}\right)\omega_b^{-nk}\right).$$

In the case that $n = 0$, the expression reduces to $\frac{1}{b}\sum_{k=1}^{b-1}\left(\frac{k}{b} - \frac{1}{2}\right) = \frac{1}{b}\left(\frac{(b-1)b}{2b} - \frac{b-1}{2}\right) = 0$, which matches the given expression at $n = 0$ as desired. Otherwise, we split the sum into two parts. We have

$$\sum_{k=1}^{b-1} -\frac{1}{2}\omega_b^{-nk} = \sum_{k=0}^{b-1} -\frac{1}{2}\omega_b^{-nk} - \left(-\frac{1}{2}\right)\omega_b^0 = \frac{1}{2}$$

and

$$\frac{1}{b}\sum_{k=1}^{b-1}\frac{k}{b}\omega_b^{-nk} = \frac{1}{b}\sum_{k=1}^{b-1}\sum_{j=k}^{b-1}\frac{1}{b}\omega_b^{-nj} = \frac{1}{b^2}\sum_{k=1}^{b-1}\frac{\omega_b^{-nk} - \omega_b^{-nb}}{1-\omega_b^{-n}} = \frac{1}{b^2}\sum_{k=1}^{b-1}\frac{\omega_b^{-nk} - 1}{1-\omega_b^{-n}} = \frac{1}{b^2}\cdot\frac{-b}{1-\omega_b^{-n}}$$

$$\frac{1}{b}\sum_{k=1}^{b-1}\frac{k}{b}\omega_b^{-nk} = \frac{1}{b}\cdot\frac{\omega_b^n}{1-\omega_b^n}.$$

Adding the two parts together gives the desired answer of $\frac{1}{2b}\cdot\frac{1+\omega_b^n}{1-\omega_b^n}$. □

Question 3.10 (10 pts)**.** Show that

$$s(a,b) = \frac{1}{4b}\sum_{m=1}^{b-1}\cot\frac{\pi m a}{b}\cot\frac{\pi m}{b}.$$

Solution 3.10. For the scaled function $B_b(an)$, we substitute into the Fourier representation

$$B_b(an) = \sum_{k=0}^{b-1} \hat{B}_b(k)\omega_b^{kan}.$$

By performing a change of variables $m = ka \bmod n$ and noticing that as k ranges from 0 to $b-1$ (since a is coprime to b), we also have that m ranges from 0 to $b-1$. Hence, letting a^{-1} denote the multiplicative inverse of a modulo b, we have that

$$B_b(an) = \sum_{m=0}^{b-1} \hat{B}_b(ma^{-1})\omega_b^{mn}.$$

Applying Parseval's identity, we have that

$$\begin{aligned}
s(a,b) &= \sum_{n=0}^{b-1} B_b(an)B_b(n) \\
&= b\sum_{m=0}^{b-1} \hat{B}_b(ma^{-1})\hat{B}_b(m) \\
&= b\sum_{m=1}^{b-1} \left(\frac{i}{2b}\cot\frac{\pi m a^{-1}}{b}\right)\left(-\frac{i}{2b}\cot\frac{\pi m}{b}\right) \\
s(a,b) &= \frac{1}{4b}\sum_{m=1}^{b-1} \cot\frac{\pi m a^{-1}}{b}\cot\frac{\pi m}{b}.
\end{aligned}$$

By reindexing the previous summation with $ka = m$, we have

$$s(a,b) = \frac{1}{4b}\sum_{k=1}^{b-1} \cot\frac{\pi k}{b}\cot\frac{\pi k a}{b},$$

as desired. □

Question 3.11 (10 pts)**.** We can relate the generalized Fourier-Dedekind sum back to the classical Dedekind sum. Given the identity

$$\frac{1}{a}\sum_{k=1}^{a-1} \frac{1}{1-\omega_a^k}\omega_a^{-kn} = \left\lfloor \frac{n}{a} \right\rfloor - \frac{n}{a} + \frac{1}{2} - \frac{1}{2a},$$

show that

$$\sigma_0(1,a;b) = \frac{1}{4} - \frac{1}{4b} - s(a,b).$$

Solution 3.11. Using Definition 3.6, we have

$$\begin{aligned}\sigma_0(a,1;b) &= \frac{1}{b}\sum_{k=1}^{b-1}\frac{1}{(1-\xi_b^{ka})(1-\xi_b^{k})}\\ &= \frac{1}{b}\sum_{k=1}^{b-1}\left(\frac{1}{1-\xi_b^{ka}}-\frac{1}{2}\right)\left(\frac{1}{1-\xi_b^{k}}-\frac{1}{2}\right)+\frac{1}{2b}\sum_{k=1}^{b-1}\frac{1}{1-\xi_b^{k}}+\frac{1}{2b}\sum_{k=1}^{b-1}\frac{1}{1-\xi_b^{ka}}-\frac{1}{b}\sum_{k=1}^{b-1}\frac{1}{4}\\ \sigma_0(a,1;b) &= \frac{1}{4b}\sum_{k=1}^{b-1}\left(\frac{1+\xi_b^{ka}}{1-\xi_b^{ka}}\right)\left(\frac{1+\xi_b^{k}}{1-\xi_b^{k}}\right)+\frac{1}{b}\sum_{k=1}^{b-1}\frac{1}{1-\xi_b^{k}}-\frac{b-1}{4b}.\end{aligned}$$

In the last step, we used the fact that multiplying the index k by a does not change the middle sum. This middle sum can be further simplified, yielding

$$\frac{1}{b}\sum_{k=1}^{b-1}\frac{1}{(1-\xi_b^{k})\xi_b^{kn}} = -\left\{\frac{n}{b}\right\}+\frac{1}{2}-\frac{1}{2b}.$$

Therefore, using the result of Question 3.10,

$$\begin{aligned}\sigma_0(a,1;b) &= \frac{1}{4b}\sum_{k=1}^{b-1}\left(\frac{1+\xi_b^{ka}}{1-\xi_b^{ka}}\right)\left(\frac{1+\xi_b^{k}}{1-\xi_b^{k}}\right)+\frac{1}{2}-\frac{1}{2b}-\frac{b-1}{4b}\\ &= \frac{1}{4b}\sum_{k=1}^{b-1}\left(i\cot\frac{\pi ka}{b}\right)\left(i\cot\frac{\pi k}{b}\right)+\frac{b-1}{4b}\\ &= -\frac{1}{4b}\sum_{k=1}^{b-1}\cot\frac{\pi ka}{b}\cot\frac{\pi k}{b}+\frac{b-1}{4b}\\ \sigma_0(a,1;b) &= \frac{1}{4}-\frac{1}{4b}-s(a,b),\end{aligned}$$

as desired. □

3.3 Applications to Point–Counting

Question 3.12 (10 pts)**.** Show that $\mathrm{Vol}(P_t)$ is always a polynomial in t. Compute the degree of this polynomial.

Solution 3.12. The polytope $P_t = \{x_i \geq 0 : a_1x_1 + \cdots + a_dx_d = t\}$ is a dilation by a factor of t of the simplex defined by $a_1x_1 + \cdots + a_dx_d = 1$. We know it is $d-1$ dimensional, so $\mathrm{Vol}(P_t) = t^{d-1}Vol(P_1)$. Since $\mathrm{Vol}(P_1)$ is a constant independent of t, the expression is a polynomial in t with degree $\boxed{d-1}$. □

Question 3.13 (5 pts)**.** Compute $\mathrm{Vol}(P_t)$ when $d = 2$.

Solution 3.13. This is just the distance between the line segment between $\left(\frac{t}{a_1}, 0\right)$ and $\left(0, \frac{t}{a_2}\right)$, which is $\boxed{\sqrt{\left(\frac{t}{a_1}\right)^2+\left(\frac{t}{a_2}\right)^2}}$.

Question 3.14 (20 pts)**.** Prove that for $d = 2$ and coprime a_1, a_2, the number of ways to make change is:

$$p_2(t) = \frac{t + \frac{a_1+a_2}{2}}{a_1 a_2} + \sigma_{-t}(a_1; a_2) + \sigma_{-t}(a_2; a_1).$$

Solution 3.14. Geometrically, $p_2(t)$ is the number of lattice points on the line segment defined by $a_1x + a_2y = t$ where $x \geq 0$ and $y \geq 0$. Because $\gcd(a_1, a_2) = 1$, by Bezout's lemma we know there exists at least one integer lattice point on the infinite line. Let (x_0, y_0) be one such solution. All integer points on this line take the parametric form

$$x = x_0 + ka_2, \qquad y = y_0 - ka_1$$

for $k \in \mathbb{Z}$. For these points to lie on our specific line segment in the first quadrant, we must enforce the boundaries $x \geq 0$ and $y \geq 0$, yielding

$$x_0 + ka_2 \geq 0 \implies k \geq -\frac{x_0}{a_2} \qquad y_0 - ka_1 \geq 0 \implies k \leq \frac{y_0}{a_1}.$$

The value $p_2(t)$ is simply the geometric count of valid integers k that fall within this closed interval $\left[-\frac{x_0}{a_2}, \frac{y_0}{a_1}\right]$. The number of integers in any interval $[L, R]$ is given by $\lfloor R \rfloor - \lceil L \rceil + 1$. Assuming t is not a multiple of a_1 or a_2 (meaning the endpoints are not integers), we can rewrite the floor and ceiling functions using the sawtooth function defined earlier as $((z)) = z - \lfloor z \rfloor - \frac{1}{2}$. For the right bound, we have

$$\left\lfloor \frac{y_0}{a_1} \right\rfloor = \frac{y_0}{a_1} - \left(\left(\frac{y_0}{a_1}\right)\right) - \frac{1}{2}.$$

For the left bound (using the ceiling identity $\lceil L \rceil = -\lfloor -L \rfloor$), we have

$$\left\lceil -\frac{x_0}{a_2} \right\rceil = -\left\lfloor \frac{x_0}{a_2} \right\rfloor = -\frac{x_0}{a_2} + \left(\left(\frac{x_0}{a_2}\right)\right) + \frac{1}{2}.$$

Thus,

$$\begin{aligned}
p_2(t) &= \left\lfloor \frac{y_0}{a_1} \right\rfloor - \left\lceil -\frac{x_0}{a_2} \right\rceil + 1 \\
&= \left(\frac{y_0}{a_1} - \left(\left(\frac{y_0}{a_1}\right)\right) - \frac{1}{2}\right) - \left(-\frac{x_0}{a_2} + \left(\left(\frac{x_0}{a_2}\right)\right) + \frac{1}{2}\right) + 1 \\
&= \frac{a_1x_0 + a_2y_0}{a_1a_2} - \left(\left(\frac{y_0}{a_1}\right)\right) - \left(\left(\frac{x_0}{a_2}\right)\right) \\
p_2(t) &= \frac{t}{a_1a_2} - \left(\left(\frac{y_0}{a_1}\right)\right) - \left(\left(\frac{x_0}{a_2}\right)\right)
\end{aligned}$$

where the first term represents the continuous 1D volume of the segment normalized to the scales a_1 and a_2. The remaining terms $-\left(\left(\frac{y_0}{a_1}\right)\right) - \left(\left(\frac{x_0}{a_2}\right)\right)$ evaluate the fractional parts at the boundaries. Using the finite Fourier series properties of the sawtooth function, we claim these terms are algebraically equivalent to the Fourier-Dedekind sums σ_{-t} with a slight offset.

$$-\left(\left(\frac{y_0}{a_1}\right)\right) = \sigma_{-t}(a_2; a_1) + \frac{1}{2a_1} \qquad -\left(\left(\frac{x_0}{a_2}\right)\right) = \sigma_{-t}(a_1; a_2) + \frac{1}{2a_2}$$

Solution 3.14. (cont.) We will do the first one and the other will follow by symmetry. By writing the sawtooth function as a Fourier series, we see that

$$-\left(\left(\frac{y_0}{a_1}\right)\right) = -\frac{1}{2a_1}\sum_{k=1}^{a_1-1}\frac{1+\omega_{a_1}^k}{1-\omega_{a_1}^k}\omega_{a_1}^{ky_0}.$$

Since $\gcd(a_1, a_2) = 1$, as a new index m ranges from 1 to $a_1 - 1$, the multiples $k \equiv ma_2 \pmod{a_1}$ perfectly permute the non-zero residues modulo a_1. We can safely re-index the sum by substituting $k = ma_2$, yielding

$$-\left(\left(\frac{y_0}{a_1}\right)\right) = -\frac{1}{2a_1}\sum_{m=1}^{a_1-1}\frac{1+\omega_{a_1}^{ma_2}}{1-\omega_{a_1}^{ma_2}}\omega_{a_1}^{ma_2y_0}.$$

From our original line equation, we established that $a_1x_0 + a_2y_0 = t$, which implies $a_2y_0 \equiv t \pmod{a_1}$. Therefore, we can substitute $\omega_{a_1}^{ma_2y_0} = \omega_{a_1}^{mt}$ to obtain

$$-\left(\left(\frac{y_0}{a_1}\right)\right) = -\frac{1}{2a_1}\sum_{m=1}^{a_1-1}\frac{1+\omega_{a_1}^{ma_2}}{1-\omega_{a_1}^{ma_2}}\omega_{a_1}^{mt}.$$

To match the negative exponent $-t$ found in the definition of the Fourier-Dedekind sum σ_{-t}, we reverse the index of summation by replacing m with $a_1 - m$ (which is equivalent to mapping $m \mapsto -m \pmod{a_1}$). This inverts all the roots of unity, yielding

$$\begin{aligned}
-\left(\left(\frac{y_0}{a_1}\right)\right) &= -\frac{1}{2a_1}\sum_{m=1}^{a_1-1}\frac{1+\omega_{a_1}^{-ma_2}}{1-\omega_{a_1}^{-ma_2}}\omega_{a_1}^{-mt} \\
&= \frac{1}{2a_1}\sum_{m=1}^{a_1-1}\frac{1+\omega_{a_1}^{ma_2}}{1-\omega_{a_1}^{ma_2}}\omega_{a_1}^{-mt} \\
&= \frac{1}{2a_1}\sum_{m=1}^{a_1-1}\left(\frac{2}{1-\omega_{a_1}^{ma_2}}-1\right)\omega_{a_1}^{-mt} \\
-\left(\left(\frac{y_0}{a_1}\right)\right) &= \frac{1}{a_1}\sum_{m=1}^{a_1-1}\frac{\omega_{a_1}^{-mt}}{1-\omega_{a_1}^{ma_2}} - \frac{1}{2a_1}\sum_{m=1}^{a_1-1}\omega_{a_1}^{-mt}.
\end{aligned}$$

The first term is exactly the definition of the Fourier-Dedekind sum $\sigma_{-t}(a_2; a_1)$. For the second term, assuming t is not a multiple of a_1 (which is consistent with our assumption that the endpoints of the line segment are not integers), the sum of all a_1-th roots of unity is 0, meaning the sum of the non-zero powers evaluates to -1. Applying this to the second term, we see that

$$-\left(\left(\frac{y_0}{a_1}\right)\right) = \sigma_{-t}(a_2; a_1) - \frac{1}{2a_1}(-1) = \sigma_{-t}(a_2; a_1) + \frac{1}{2a_1}.$$

By symmetry, swapping the indices yields

$$-\left(\left(\frac{x_0}{a_2}\right)\right) = \sigma_{-t}(a_1; a_2) + \frac{1}{2a_2}.$$

Substituting these into our geometric count gives

$$\begin{aligned}
p_2(t) &= \frac{t}{a_1a_2} - \left(\left(\frac{y_0}{a_1}\right)\right) - \left(\left(\frac{x_0}{a_2}\right)\right) = \frac{t}{a_1a_2} + \sigma_{-t}(a_2; a_1) + \frac{1}{2a_1} + \sigma_{-t}(a_1; a_2) + \frac{1}{2a_2} \\
p_2(t) &= \frac{t + \frac{a_1+a_2}{2}}{a_1a_2} + \sigma_{-t}(a_1; a_2) + \sigma_{-t}(a_2; a_1),
\end{aligned}$$

as desired. □

Question 3.15 (10 pts). Express $p_d^\circ(t)$ in terms of $p_d(t)$.

Solution 3.15. The value $p_d^\circ(t)$ counts the number of internal lattice points, i.e., the number of integer solutions $(x_1, x_2, \ldots, x_d)$ to $\sum_{i=1}^{d} a_i x_i = t$, where $x_i > 0$ for all i. We can perform a change of variables. Let $y_i = x_i - 1$. Since $x_i \geq 1$ are integers, it follows that $y_i \geq 0$. Substituting $x_i = y_i + 1$ into our constraint gives

$$\sum_{i=1}^{d} a_i(y_i + 1) = t \implies \sum_{i=1}^{d} a_i y_i = t - \sum_{i=1}^{d} a_i.$$

The number of non-negative integer solutions $(y_1, y_2, \ldots, y_d)$ to this new equation is exactly $p_d\left(t - \sum_{i=1}^{d} a_i\right)$.

Thus, there is a bijection between the sets of solutions, and $\boxed{p_d^\circ(t) = p_d\left(t - \sum_{i=1}^{d} a_i\right)}$.

Question 3.16 (10 pts). Prove the **Ehrhart-Macdonald Reciprocity** law holds for this polytope with $d = 2$:

$$p_2^\circ(t) = -p_2(-t).$$

Do not use the general reciprocity theorem for this question.

Solution 3.16. From Question 3.15, we established the bijection $p_2^\circ(t) = p_2(t - a_1 - a_2)$ for $d = 2$. Substituting $t - a_1 - a_2$ into our formula for $p_2(t)$ from Question 3.14 gives

$$p_2^\circ(t) = \frac{(t - a_1 - a_2) + \frac{a_1 + a_2}{2}}{a_1 a_2} + \sigma_{-(t-a_1-a_2)}(a_1; a_2) + \sigma_{-(t-a_1-a_2)}(a_2; a_1)$$
$$p_2^\circ(t) = \frac{t - \frac{a_1 + a_2}{2}}{a_1 a_2} + \sigma_{-(t-a_1-a_2)}(a_1; a_2) + \sigma_{-(t-a_1-a_2)}(a_2; a_1).$$

Next, we analyze the Fourier-Dedekind sums. Since the second argument of $\sigma_n(a_1; a_2)$ is evaluated modulo a_2 (since that's the base of all the roots of unity), the shift by $-a_2$ in the index disappears, leaving

$$\sigma_{-(t-a_1-a_2)}(a_1; a_2) = \sigma_{-t+a_1}(a_1; a_2).$$

Expanding this into the form of Definition 3.6, we have

$$\sigma_{-t+a_1}(a_1; a_2) = \frac{1}{a_2} \sum_{k=1}^{a_2-1} \frac{\omega_{a_2}^{(-t+a_1)k}}{1 - \omega_{a_2}^{ka_1}} = \frac{1}{a_2} \sum_{k=1}^{a_2-1} \omega_{a_2}^{-tk} \frac{\omega_{a_2}^{ka_1}}{1 - \omega_{a_2}^{ka_1}}.$$

To match the terms in $p_2(-t)$, we re-index the sum by substituting $k = a_2 - j$ (which maps $k \mapsto -j \pmod{a_2}$), yielding

$$\sigma_{-t+a_1}(a_1; a_2) = \frac{1}{a_2} \sum_{j=1}^{a_2-1} \omega_{a_2}^{-t(-j)} \frac{\omega_{a_2}^{-ja_1}}{1 - \omega_{a_2}^{-ja_1}} = \frac{1}{a_2} \sum_{j=1}^{a_2-1} \omega_{a_2}^{tj} \frac{\omega_{a_2}^{-ja_1} \cdot \omega_{a_2}^{ja_1}}{(1 - \omega_{a_2}^{-ja_1})\omega_{a_2}^{ja_1}}$$
$$= \frac{1}{a_2} \sum_{j=1}^{a_2-1} \omega_{a_2}^{tj} \frac{1}{\omega_{a_2}^{ja_1} - 1} = -\frac{1}{a_2} \sum_{j=1}^{a_2-1} \omega_{a_2}^{tj} \frac{1}{1 - \omega_{a_2}^{ja_1}}$$
$$\sigma_{-t+a_1}(a_1; a_2) = -\sigma_t(a_1; a_2).$$

Solution 3.16. (cont.) By symmetry, the other sum yields $\sigma_{-(t-a_1-a_2)}(a_2; a_1) = -\sigma_t(a_2; a_1)$. Therefore,

$$\begin{aligned} p_2^\circ(t) &= \frac{t - \frac{a_1+a_2}{2}}{a_1a_2} + \sigma_{-(t-a_1-a_2)}(a_1; a_2) + \sigma_{-(t-a_1-a_2)}(a_2; a_1) \\ &= \frac{t - \frac{a_1+a_2}{2}}{a_1a_2} - \sigma_t(a_1; a_2) - \sigma_t(a_2; a_1) \\ &= -\left(\frac{-t + \frac{a_1+a_2}{2}}{a_1a_2} + \sigma_{-(-t)}(a_1; a_2) + \sigma_{-(-t)}(a_2; a_1) \right) \\ p_2^\circ(t) &= -p_2(-t), \end{aligned}$$

as desired. [Since this polytope is a simplex with dimension $d-1$, by Ehrhart-Macdonald Reciprocity (Theorem 2.9), we expect $p_2^\circ(t) = (-1)^{2-1}p_2(-t) = -p_2(-t)$ at $d = 2$.] □

ERRATUM: During the competition, this question had the wrong sign on $p_2(-t)$. We adjusted credit accordingly to account for the error.

3.4 Higher–Dimensional Point-Counting

Question 3.17 (5 pts)**.** Explain briefly why the assumption that $\gcd(a_j, a_k) = 1$ for all $j \neq k$ implies that all poles besides $z = 1$ have order 1.

Solution 3.17. The generating function $F(z)$ has a denominator containing the factors $(1 - z^{a_j})$. The poles occur at the roots of unity, specifically the a_j-th roots of unity for each j. A pole λ will have an order greater than 1 if and only if it is a root of the denominator multiple times, meaning it must be an a_j-th root of unity and an a_k-th root of unity for some $j \neq k$. If λ is both, its order must divide both a_j and a_k. However, we are given that $\gcd(a_j, a_k) = 1$ for all $j \neq k$. The only integer dividing both is 1, which corresponds to the root $\lambda = 1$ (the pole at $z = 1$). Thus, any root $\lambda \neq 1$ cannot be shared between factors, meaning all poles besides $z = 1$ have an algebraic multiplicity of 1 (simple poles).

Question 3.18 (10 pts)**.** Fix an index $j \in \{1, \ldots, d\}$ and a specific exponent $k \in \{1, \ldots, a_j - 1\}$ and let $\lambda = \omega_{a_j}^k$. We want to find the coefficient $A_{j,k}$ in the term $\frac{A_{j,k}}{1-\lambda z}$. Using the residue formula for a simple pole, show that:

$$A_{j,k} = \frac{1}{a_j} \prod_{m \neq j} \frac{1}{1 - \omega_{a_j}^{-ka_m}}.$$

Solution 3.18. We are given the term $\frac{A_{j,k}}{1-\lambda z}$ in the partial fraction expansion, where $\lambda = \omega_{a_j}^k$. This corresponds to a simple pole at $z = \lambda^{-1} = \omega_{a_j}^{-k}$. We can extract the coefficient $A_{j,k}$ by multiplying the generating function by $(1-\lambda z)$ and taking the limit as $z \to \lambda^{-1}$, yielding

$$A_{j,k} = \lim_{z\to\lambda^{-1}}(1-\lambda z)F(z) = \lim_{z\to\lambda^{-1}} \frac{1-\lambda z}{1-z^{a_j}} \prod_{m\neq j} \frac{1}{1-z^{a_m}}.$$

We evaluate the first fraction using L'Hôpital's rule, taking the derivative with respect to z. We obtain

$$\lim_{z\to\lambda^{-1}} \frac{-\lambda}{-a_j z^{a_j-1}} = \frac{-\lambda}{-a_j(\lambda^{-1})^{a_j-1}} = \frac{\lambda^{a_j}}{a_j}.$$

Since $\lambda = \omega_{a_j}^k$, it is an a_j-th root of unity, meaning $\lambda^{a_j} = 1$. Thus, the limit of the first fraction is $\frac{1}{a_j}$. Evaluating the remaining product at $z = \omega_{a_j}^{-k}$ yields

$$A_{j,k} = \frac{1}{a_j} \prod_{m\neq j} \frac{1}{1-\omega_{a_j}^{-ka_m}},$$

as desired. □

Question 3.19 (15 pts)**.** The term derived above contributes $A_{j,k}\lambda^t$ to the coefficient of z^t in the power series expansion of $F(z)$. Show that

$$p_d(t) = \mathrm{Poly}_d(t) + \sum_{j=1}^{d} \sigma_{-t}(a_1,\ldots,a_{j-1},a_{j+1},\ldots,a_d;a_j),$$

where $\mathrm{Poly}_d(t)$ is a polynomial of degree $d-1$ (you need not derive an explicit form for this polynomial).

Solution 3.19. The partial fraction decomposition of $F(z)$ is given as the sum of the pole at $z = 1$ and the sum over other roots of unity. The pole at $z = 1$ has order d. from Question 2.2, the terms $\frac{C_m}{(1-z)^m}$ expand as $C_m \sum_{t=0}^{\infty} \binom{t+m-1}{m-1} z^t$. Since $\binom{t+m-1}{m-1}$ is a polynomial in t of degree $m-1$, the sum of these terms for $m = 1, 2, \ldots, d$ yields $\mathrm{Poly}_d(t)$, a polynomial of degree $d-1$. For the remaining simple poles $\lambda = \omega_{a_j}^k$, the term $\frac{A_{j,k}}{1-\lambda z}$ contributes $A_{j,k}\lambda^t$ to the coefficient of z^t. Summing over all $j \in \{1, 2, \ldots, d\}$ and all $k \in \{1, 2, \ldots, a_j - 1\}$ yields

$$\sum_{j=1}^{d} \sum_{k=1}^{a_j-1} A_{j,k}(\omega_{a_j}^k)^t = \sum_{j=1}^{d} \frac{1}{a_j} \sum_{k=1}^{a_j-1} \frac{\omega_{a_j}^{kt}}{\prod_{m\neq j}(1-\omega_{a_j}^{-ka_m})}.$$

Comparing this to Definition 3.6, and noting that $\omega^{-ka_m} = \omega^{k(-a_m)}$, this is precisely $\sum_{j=1}^{d} \sigma_{-t}(a_1,\ldots,a_{j-1},a_{j+1},\ldots,a_d;a_j)$. Therefore,

$$p_d(t) = \mathrm{Poly}_d(t) + \sum_{j=1}^{d} \sigma_{-t}(a_1,\ldots,a_{j-1},a_{j+1},\ldots,a_d;a_j),$$

as desired. □