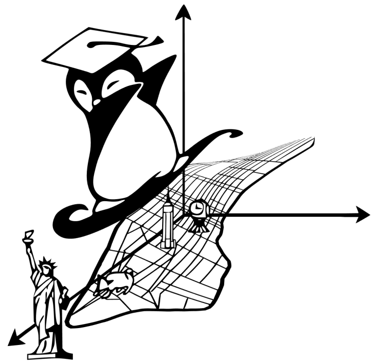
ICMT — Individual Round (Division B)



Do not flip open this packet until instructed by your proctor.

Round Structure

- This is an **individual** test. You will have **75 minutes** total.
- There are **12 problems** total. Problems 8 and 11 are *estimation* problems; you will get more points for an answer *closer* to the exact answer. The remaining 10 problems are standard short answer problems.
- Problems are roughly ordered by difficulty.

Scoring

- The 10 standard short answer problems are worth 1 point for a correct answer, and 0 points for an incorrect answer.
- The 2 estimation problems have separate scoring systems, which will be explained in their respective problem statements. Each estimation problem will give at least 0 points and at most 1 point.
- Your total score on the round will be the sum of the points scored on all 12 problems.
- In case of a tie, we will implement the following tiebreaker system:
  1. First, the 10 short answer problems are ordered from hardest to easiest, based on the number of correct submissions.
     - The estimation problems will not be taken into account for the tiebreaker system.
     - In case of short answer problems having an equal number of solves, the problems will then be ordered from later in the test to earlier.
  2. The n^{th} hardest problem is assigned a tiebreaking value of 2^{-n}. For example, the hardest problem will have a tiebreaking value of $\frac{1}{2}$, the second hardest problem will have a tiebreaking value of $\frac{1}{4}$, et cetera.
  3. Each student's tiebreaking index is calculated as the sum of their original score and the tiebreaker values of the problems that they correctly answered.
  4. Students are ordered by their tiebreaking index, determining their tiebroken rank.

  Roughly speaking, among those tied for the same score, whoever solved the hardest short answer problem is placed the highest, followed by the one who solved the next hardest, and so on.

1. Define $f(x) = x^3 - 2x + 1$. What is the sum of all of the complex roots of f, f', f'', and f''', counted with multiplicity (i.e. summing double roots twice, triple roots three times, etc.)?

2. Triangle $\triangle ABC$ is drawn in the (x, y)-plane with points $A = (5, 0)$, $B = (0, 5)$, and $C = (1, 7)$. Find the area of the region swept out by $\triangle ABC$ when it is rotated 360° around the origin $(0, 0)$.

3. Suppose a real analytic function f satisfies $f(20) = 0$ and $f^{(n)}(20) = n$ for all positive integers n, where $f^{(n)}$ is the n^{th} derivative of f. What is $f(26)$?

4. For a nonconstant real polynomial f, define $L(f)$ to be the minimum possible number of distinct real solutions to $f(x) = c$ over all c in the range of f. Let M be the maximum possible value of $L(f)$ over all polynomials f, and let d be the minimum possible degree of a polynomial f such that $L(f) = M$. What is $M + d$?

5. A segment $\overline{AB}$ in the plane has length 6. Let R be the set of all possible points C in the same plane such that $\triangle ABC$ is an acute triangle whose longest side is $\overline{AB}$. Find the area of R.

6. Let M be the maximum possible number of distinct real roots of the polynomial $f_n(x) = (x^n - n)^2 + n(x^n - 20)$ over all positive integers n. Determine the sum of all positive integers n for which f_n has M distinct real roots.

7. Rohit is sorting 15 distinguishable pairs of socks in a pile. Rohit takes socks one at a time uniformly at random from the pile. If he isn't holding the other sock in the pair, he holds onto the sock; if he is, he pairs both socks and sets them aside. After Rohit has taken 15 socks in total, what is the expected number of socks Rohit is holding?

8. Estimate, as a positive integer, the number of positive integers $n < 10^{1000}$ with the property that if n is k digits long, the last k digits of n^3 are equal to n. If your estimate is E and the correct answer A, you will receive $\max\left(0, 1 - 2\ln\max\left(\frac{A}{E}, \frac{E}{A}\right)\right)$ points for a valid estimate, and 0 points for an invalid estimate.

9. Let $\mathcal{F}$ be the set of monic (leading coefficient 1) polynomials with complex coefficients which divide the polynomial $x^5 + 2x^4 + 3x^2 + 4x + 5$. Compute
$$\sum_{f \in \mathcal{F}} f(1).$$

10. Consider the sum
$$S(N) = \sum_{1 \le a, b \le N} \frac{1}{a + bi}.$$
There exist positive real numbers c, d such that $\lim_{N \to \infty} \frac{|S(N)|}{N^d} = c$. Compute c.

11. Alice and Bob are playing a game with n stones. They alternate turns with Alice starting, and each turn, someone takes a number of stones from the pile that is a divisor of the total number of stones. The person to take the last stone loses. For a pile of n stones, let $S(n)$ be the set of all possible numbers of stones that Alice may take in her first move so that she wins with perfect play, and let $A(n)$ be the sum of all numbers in $S(n)$. Estimate, as a positive integer, the value of
$$\sum_{n=1}^{4000} A(n).$$
If your estimate is E and the correct answer A, you will receive $\max\left(0, 3 - 2\max\left(\frac{A}{E}, \frac{E}{A}\right)\right)$ points for a valid estimate, and 0 points for an invalid estimate.

12. A point P is selected uniformly at random from the surface of a sphere centered at $O = (2, 0, 0)$ with radius 2 (so that a point is equally likely to be selected from two regions of equal area on the surface). The segment ℓ is drawn with P as an endpoint and O as its midpoint. Let $S(\ell)$ be the surface created by revolving ℓ about the y-axis, let M be the maximum y-coordinate of any point in ℓ, and let $V(\ell)$ be the volume of the solid enclosed by $S(\ell)$ and the planes $y = \pm M$. Find the expected value of $V(\ell)$.