ICMT — Individual Round Solutions (Division B)

1. Define $f(x) = x^3 - 2x + 1$. What is the sum of all of the complex roots of f, f', f'', and f''', counted with multiplicity (i.e. summing double roots twice, triple roots three times, etc.)?

   Answer: 0

   Solution: We can do this manually: $f'(x) = 3x^2 - 2$, $f''(x) = 6x, f'''(x) = 6$, then we use standard factoring methods. (For the cubic, we can notice that $x = 1$ is a root and then factor the remaining quadratic.) We can speed this up by using one of Vieta's formulas: the sum of the roots is the negative of the second coefficient over the leading coefficient. However, the second coefficient is always 0; this is true in f, and the derivative preserves this property (as it decreases all powers by 1 simultaneously), so the sum of all roots must be $\boxed{0}$.

2. Triangle $\triangle ABC$ is drawn in the (x, y)-plane with points $A = (5, 0)$, $B = (0, 5)$, and $C = (1, 7)$. Find the area of the region swept out by $\triangle ABC$ when it is rotated 360° around the origin $(0, 0)$.

   Answer: $\frac{75\pi}{2}$

   Solution: Since we are doing a full rotation around the origin, the region swept out will be a ring around the origin. The radius of the inner circle will be the least distance from any point of the triangle to the origin, and the radius of the outer circle will be the greatest distance of any point of the triangle to the origin. The farthest point is C at a distance of $\sqrt{50}$, and the closest point is the midpoint of $\overline{AB}$ at a distance of $\frac{5}{\sqrt{2}}$ (using 45-45-90 triangles). So the area of the ring is

$$\pi(\sqrt{50})^2 - \pi\left(\frac{5}{\sqrt{2}}\right)^2 = 50\pi - \frac{25\pi}{2} = \boxed{\frac{75\pi}{2}}.$$

3. Suppose a real analytic function f satisfies $f(20) = 0$ and $f^{(n)}(20) = n$ for all positive integers n, where $f^{(n)}$ is the n^{th} derivative of f. What is $f(26)$?

   Answer: $6e^6$

   Solution: We do a Taylor series expansion at $x = 20$, getting

$$f(x) = \sum_{n=0}^{\infty} \frac{n}{n!}(x-20)^n = \sum_{n=1}^{\infty} \frac{n}{n!}(x-20)^n$$

   since the first term is 0. Then we can divide out n to get

$$\sum_{n=1}^{\infty} \frac{1}{(n-1)!}(x-20)^n = (x-20) \cdot \sum_{n=0}^{\infty} \frac{(x-20)^n}{n!}$$

   By the Taylor series for $e^x = \sum_{n=0}^{\infty} x^n/n!$, we get that $f(x) = (x - 20) \cdot e^{x-20}$, and therefore plugging in $x = 26$ gives $\boxed{6e^6}$.

4. For a nonconstant real polynomial f, define $L(f)$ to be the minimum possible number of distinct real solutions to $f(x) = c$ over all c in the range of f. Let M be the maximum possible value of $L(f)$ over all polynomials f, and let d be the minimum possible degree of a polynomial f such that $L(f) = M$. What is $M + d$?

   Answer: 6

   Solution: We start with the fact that a polynomial function can only have finitely many points where its derivative is 0 (since its derivative is also a polynomial). So, the function can only change direction a finite number of times. Therefore, we can divide f into a monotone left piece that goes to $\pm\infty$, a bounded finite interval, and a monotone right piece that goes to $\pm\infty$. Because the left and right pieces are both monotone, any horizontal line can only intersect them at most once. Then, since the pieces are unbounded, there exists some point on a piece that is outside the range of f on the bounded interval.

A horizontal line through that point can intersect at most the left and right pieces of the function, which by our earlier discussion means that such a line can intersect at most two points on f, which limits $L(f)$ to 2.

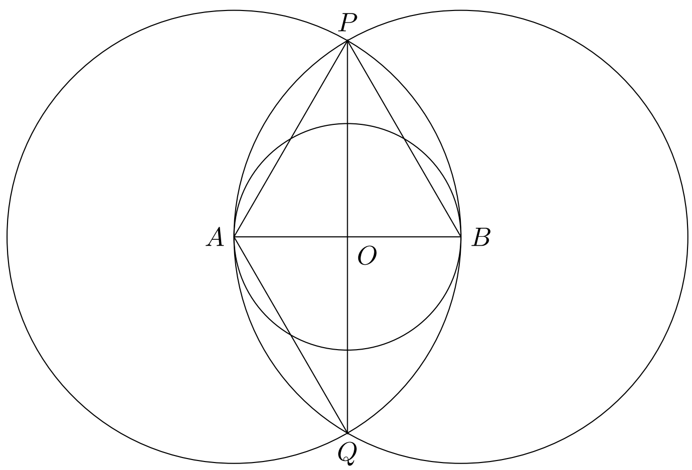
To see that $L(f) = 2$ is possible, we start by considering the function $y = x^2$, which almost works except for the fact that the parabola has a vertex at $(0, 0)$, and so only one intersection with $y = 0$. To fix this, we can use $y = (x^2 - 1)^2$, which will "fold over" the single vertex of $x^2 - 1$ at $(0, -1)$ so that every horizontal line hits at least two points.

Then, by this analysis, it becomes clear that the minimal degree of a working function is 4: odd degree functions all have $L(f) = 1$ (because the left and right pieces go in opposite directions) and parabolas have $L(f) = 1$ because of their vertices. This gives an answer of $2 + 4 = \boxed{6}$.

5. A segment $\overline{AB}$ in the plane has length 6. Let R be the set of all possible points C in the same plane such that $\triangle ABC$ is an acute triangle whose longest side is $\overline{AB}$. Find the area of R.

Answer: $15\pi - 18\sqrt{3}$

Solution:



Draw circle ω_A with center A and radius AB, circle ω_B with center B and radius AB, and circle ω with diameter $\overline{AB}$ and center O (the midpoint of $\overline{AB}$). Note that if C lies outside ω_A, then $AB < AC$, and if C lies outside of ω_B, then $AB < BC$. Therefore, C must lie in the intersection of ω_A and ω_B.

Then, consider what happens when C lies on the perimeter of ω. By the inscribed angle theorem, $\angle ACB$ is half of $\angle AOB$. However, O lies on $\overline{AB}$, making $\measuredangle AOB = \pi$ and $\measuredangle ACB = \frac{\pi}{2}$, meaning $\triangle ABC$ is right at C. Thus, if we are inside ω, since the side lengths AC and BC will be smaller while AB stays the same, by law of cosines the angle at C will be obtuse and $\triangle ABC$ will not be acute as needed, so C must lie outside of ω.

Finally, we need to ensure that $\angle A$ and $\angle B$ are also acute; this is simpler, as we just need to restrict C to lie within the perpendiculars to $\overline{AB}$ at A and B to ensure both angles are acute, which is already forced by the previous constraints on lengths from circles ω_A, ω_B. Therefore, R is the region inside circles ω_A, ω_B but outside ω.

We need to compute the area of the intersection of ω_A and ω_B and subtract the area of ω. We can do this by subtracting the triangle from the sector and doubling. Let P and Q be the intersections of ω_A and ω_B. Then since $\triangle APB$ and $\triangle AQB$ are equilateral (all lengths are equal to AB), we have $\measuredangle PAQ = \frac{2\pi}{3}$. Therefore, half of the area of the intersection is $\frac{1}{3} \cdot \pi \cdot 6^2 - \frac{1}{2} \cdot 6\sqrt{3} \cdot 3 = 12\pi - 9\sqrt{3}$. Doubling this and subtracting the area of ω (which is 9π) gives $\boxed{15\pi - 18\sqrt{3}}$.

6. Let M be the maximum possible number of distinct real roots of the polynomial $f_n(x) = (x^n - n)^2 + n(x^n - 20)$ over all positive integers n. Determine the sum of all positive integers n for which f_n has M distinct real roots.

Answer: 72

Solution: Set $y = x^n$, and we have that $y^2 - ny + (n^2 - 20n) = 0$. This has a maximum of two real solutions for y, and $y = x^n$ has a maximum of two real solutions for x when n is even and y is positive. Therefore, there is a maximum of four real roots of $x^{2n} - nx^n + (n^2 - 20n)$. By the quadratic formula, we have

$$y = \frac{n \pm \sqrt{n^2 - (4n^2 - 80n)}}{2}.$$

We need y to take two distinct real values as solutions, so we restrict the discriminant to be positive: $n^2 - (4n^2 - 80n) = 80n - 3n^2 > 0$, which requires $n < \frac{80}{3}$. However, since n is even, we also need y to be positive in both cases for $x^n = y$ to have two real solutions. This requires $n - \sqrt{80n - 3n^2} > 0$, or $n^2 > 80n - 3n^2$ which implies $n > 20$. The final answer is the sum of the even numbers between 20 and $\frac{80}{3}$ (excluding endpoints), giving an answer of $22 + 24 + 26 = \boxed{72}$.

7. Rohit is sorting 15 distinguishable pairs of socks in a pile. Rohit takes socks one at a time uniformly at random from the pile. If he isn't holding the other sock in the pair, he holds onto the sock; if he is, he pairs both socks and sets them aside. After Rohit has taken 15 socks in total, what is the expected number of socks Rohit is holding?

Answer: $\frac{225}{29}$

Solution: Let P be the expected number of pairs of socks that Rohit pairs up after taking 15 total socks. Then the expected number of socks Rohit is holding is $15 - 2P$; it suffices to compute P.

For each of the 15 pairs of socks, the probability that Rohit pairs up the specific pair of socks is precisely the probability that both socks appear in the first 15 socks. The probability this occurs is

$$\frac{\binom{15}{2}}{\binom{30}{2}} = \frac{7}{29}.$$

Thus, by linearity of expectation, we have that

$$P = 15 \cdot \frac{7}{29} = \frac{105}{29},$$

and hence the expected number of socks Rohit is holding is

$$15 - 2 \cdot \frac{105}{29} = \boxed{\frac{225}{29}}.$$

8. Estimate, as a positive integer, the number of positive integers $n < 10^{1000}$ with the property that if n is k digits long, the last k digits of n^3 are equal to n. If your estimate is E and the correct answer A, you will receive $\max\left(0, 1 - 2\ln\max\left(\frac{A}{E}, \frac{E}{A}\right)\right)$ points for a valid estimate, and 0 points for an invalid estimate.

Answer: 11971

Solution: For a k-digit number n, it is necessary that $n^3 - n \equiv 0 \pmod{10^k}$. We factor into $n(n+1)(n-1) \equiv 0 \pmod{10^k}$. The maximum GCD of these numbers is 2 (by Euclidean algorithm). Therefore one of these three factors must have all the 5s and another (not necessarily distinct) factor must have at least all but one of the 2s; i.e., one of these three factors must be $x \cdot 5^k$, and one must be $y \cdot 2^{k-1}$. This means we need $n \equiv 0, \pm 1 \pmod{5^k}$. Modulo 2^k, we can take $0, \pm 1$, but we can also take $2^{k-1} \pm 1$ since we will get our last factor of 2 from $2^{k-1} \pm 2$. Naively, we would have 15 solutions here $\pmod{10^k}$ by combining each pair of residues with CRT, but there are some restrictions we have to impose here. If we immediately guess $15 \cdot 1000 = 15000$, we will receive 0.548 points. If we forget about the possibility of $2^{k-1} \pm 1 \mod 2^k$, we guess $9 \cdot 1000 = 9000$ and receive 0.429 points.

To improve our estimate, we must realize that we need the solutions to have k many digits mod 10^k (they need to be at least 10^{k-1}). There are several effects of this consideration. For one, we can model this as a purely uniformly random event, saying that each solution is between 10^{k-1} and 10^k with probability $\frac{10^k-10^{k-1}}{10^k} = \frac{9}{10}$. However, we also have to consider that some solutions are guaranteed to fail or succeed as they can be solved for any k identically. If we begin with our multiplication (and have correctly identified the 15 candidates), we guess $\frac{9}{10} \cdot 15000 = 13500$, which gets a score of 0.760 points.

We have some solutions that are guaranteed to fail by being too small. For one, we can't have $n \equiv 0 \pmod{5^k}$ and $0 \pmod{2^k}$ as this will be $0 \pmod{10^k}$ which has no digits. Additionally, when $n \equiv 1 \pmod{5^k}$ and $n \equiv 1 \pmod{2^k}$, this gives $n \equiv 1 \pmod{10^k}$, and 1 is not in our range from 10^{k-1} to 10^k for positive $k > 1$. We conclude that there are a total of 13 potential solutions for n modulo 10^k. Multiplying this by $\frac{9}{10}$ gives 11700, which scores 0.954 points.

We also have some solutions that are guaranteed to be large enough to have k digits. Firstly, when $n \equiv -1 \pmod{5^k}$ and $n \equiv -1 \pmod{2^k}$, this solves to $10^k - 1$ which always has k digits. Also, when $n \equiv -1 \pmod{5^k}$ and $n \equiv 2^{k-1} - 1 \pmod{2^k}$, we can note that n must be $-1 \pmod{5^k \cdot 2^{k-1} = 5 \cdot 10^{k-1}}$, and $n = 5 \cdot 10^{k-1} - 1$. Finally, when $n \equiv 1 \pmod{5^k}$ and $n \equiv 2^{k-1} + 1 \pmod{2^k}$, similarly to the previous case we have $n \equiv 1 \pmod{5^k \cdot 2^{k-1}}$, which means $n = 5 \cdot 10^{k-1} + 1$ since we already ruled out $n = 1$. These three solutions are all large enough to force k digits. Therefore, we should actually estimate $(3 + 10 \cdot \frac{9}{10}) \cdot 1000 = 12000$, getting a score of 0.995 points, a good place to stop under time constraints.

For a very modest point gain, we can calculate the small numbers $k = 1, 2$ where our analysis fails to capture the details of the situation (since there are only two residues mod 2 and four residues mod 4), getting 13 possible n less than 100. This leads us to subtract $(2 \cdot 12 - 13) = 11$ from our answer, giving the approximation 11989, which is 18 off from the true answer of $\boxed{11971}$ and scores 0.997 points.

9. Let $\mathcal{F}$ be the set of monic (leading coefficient 1) polynomials with complex coefficients which divide the polynomial $x^5 + 2x^4 + 3x^2 + 4x + 5$. Compute

$$\sum_{f \in \mathcal{F}} f(1).$$

Answer: 89

Solution: One can verify that the roots of this polynomial are distinct: taking a derivative gives $5x^4 + 8x^3 + 6x + 4$, and performing the Euclidean algorithm with f and f' gives a GCD of 1, showing no multiple roots. Factor

$$x^5 + 2x^4 + 3x^2 + 4x + 5 = \prod_{i=1}^{5} (x - \alpha_i)$$

for distinct complex numbers $\alpha_1, \ldots, \alpha_5$. Then a generic element $f \in \mathcal{F}$ looks like $\prod_{i \in S}(x - \alpha_i)$ where $S \subseteq \{1, 2, 3, 4, 5\}$, so we would like to compute the sum

$$\sum_{S \subseteq \{1,2,3,4,5\}} \left(\prod_{i \in S} (1 - \alpha_i) \right).$$

The point is that this sum factors as

$$\prod_{i=1}^{5} (1 + (1 - \alpha_i)),$$

which can be seen by a direct expansion. Of course, this is just

$$2^5 + 2 \cdot 2^4 + 3 \cdot 2^2 + 4 \cdot 2 + 5,$$

which is $\boxed{89}$.

10. Consider the sum

$$S(N) = \sum_{1 \le a,b \le N} \frac{1}{a+bi}.$$

There exist positive real numbers c, d such that $\lim_{N \to \infty} \frac{|S(N)|}{N^d} = c$. Compute c.

Answer: $\sqrt{2}\left(\frac{\ln 2}{2} + \frac{\pi}{4}\right)$

Solution: First, we rewrite the sum into a double sum:

$$S(N) = \sum_{a=1}^{N} \sum_{b=1}^{N} \frac{1}{a+bi}$$

Note that we can multiply by $\frac{1}{N}$ in the numerator and denominator, and then move the $\frac{1}{N}$ in the numerator outside the inner sum to get

$$S(N) = \sum_{a=1}^{N} \frac{1}{N} \sum_{b=1}^{N} \frac{1}{\frac{a}{N} + \frac{b}{N} i}$$

In the limit as N goes to infinity, the inner sum now becomes a Riemann sum which we can write as an integral:

$$\lim_{n \to \infty} S(N) = \sum_{a=1}^{N} \int_0^1 \frac{1}{\frac{a}{N} + yi} \, dy$$

Then, if we divide the whole sum by N, we can do the same thing again with a:

$$\lim_{n \to \infty} \frac{S(N)}{N} = \frac{1}{N} \sum_{a=1}^{N} \int_0^1 \frac{1}{\frac{a}{N} + yi} \, dy = \iint_{[0,1]^2} \frac{1}{x+yi} \, dy \, dx$$

This integral in fact converges to a finite value, despite the fact that $x + yi$ gets arbitrarily close to 0, which tells us that $|S(N)|$ is $O(N)$ in the limit. From here, there are two methods to evaluate the integral. In method 1, we can deal with i as a constant, which leads to some slightly messy computations at the end and taking logarithms of complex numbers. Alternatively in method 2, we can multiply by $\frac{x-yi}{x-yi}$ to get

$$\iint_{[0,1]^2} \frac{x-yi}{x^2+y^2} \, dy \, dx.$$

We can see that the x and y term will be equal in magnitude by symmetry, so we can simplify to taking the integral

$$\iint_{[0,1]^2} \frac{x}{x^2+y^2} \, dy \, dx.$$

and multiply by $\sqrt{2}$ in the end, which avoids complex logarithms and some of the computational mess at the cost of making the integral harder.

Method 1: We integrate first with respect to x:

$$\iint_{[0,1]^2} \frac{1}{x+yi} = \int_0^1 \ln(x+yi)|_0^1 \, dy = \int_0^1 (\ln(1+yi) - \ln yi) \, dy$$

We can evaluate these separately, using the general fact that $\int \ln x = x\ln x - x$ and some substitution:

$$\begin{aligned}
\int_0^1 (\ln(1+yi) - \ln yi)\,dy &= \frac{1}{i}\,(u\ln u - u)\,|_1^{1+i} - \frac{1}{i}\,(u\ln u - u)\,|_0^{i} \\
&= i[(i\ln i - i) - \Big(\lim_{u\to 0} u\ln u - 0\Big)] \\
&- i[(1+i)\ln(1+i) - (1+i) - (1\ln 1 - 1)] \\
&= i\left[\left(\frac{-\pi}{2} - i\right) - \left((1+i)\left(\frac{\ln 2}{2} + \frac{i\pi}{4}\right) - (1+i) + 1\right)\right] \\
&= i\left[\left(\frac{-\pi}{2} - \frac{\ln 2}{2} + \frac{\pi}{4}\right) + \left(-1 - \left(\frac{\ln 2}{2} + \frac{\pi}{4} - 1\right)\right)i\right] \\
&= -i\left[(1+i)\left(\frac{\pi}{4} + \frac{\ln 2}{2}\right)\right]
\end{aligned}$$

From here, we can compute the magnitude directly:

$$\left|-i\left[(1+i)\left(\frac{\pi}{4} + \frac{\ln 2}{2}\right)\right]\right| = |-i||1+i|\left|\frac{\pi}{4} + \frac{\ln 2}{2}\right| = \boxed{\sqrt{2}\left(\frac{\pi}{4} + \frac{\ln 2}{2}\right)}$$

Method 2: We have the integral

$$\iint_{[0,1]^2} \frac{x}{x^2+y^2}\,dy\,dx$$

We swap the integration order to allow us to substitute $u = x^2 + y^2$, $du = 2x\,dx$. This gives us an integral of the form du/u, which is $\ln u$; evaluating this gives us the integrals

$$\frac{1}{2}\left(\int_0^1 \ln\big(y^2+1\big) - 2\ln y\,dy\right)$$

The integral of $\ln y$ is $y\ln y - y$, and evaluating at the endpoints (taking the limit in the case of 0) gives us $(-1) - 0 = -1$. Plugging this back into our integral gives

$$1 + \frac{1}{2}\int_0^1 \ln\big(y^2+1\big)\,dy$$

To integrate $\ln\big(y^2+1\big)$, we use integration by parts: let $u = \ln\big(y^2+1\big)$, $dv = dy$, and we get

$$\int_0^1 \ln\big(y^2+1\big)\,dy = y\ln\big(y^2+1\big)\big|_0^1 - \int_0^1 y\cdot\frac{2y}{y^2+1}\,dy.$$

Reducing the left by plugging in the endpoints gives $\ln 2$, and on the right we have

$$-2 + 2\int_0^1 \frac{dy}{y^2+1}$$

which is the arctan integral, giving

$$-2 + 2\tan^{-1}(1) - 2\tan^{-1}(0) = -2 + \frac{\pi}{2}.$$

Putting this all together we have $1 + \frac{1}{2}\left(\ln 2 - 2 + \frac{\pi}{2}\right) = \frac{\ln 2}{2} + \frac{\pi}{4}$, which we finally multiply by $\sqrt{2}$ to get our answer of $\boxed{\sqrt{2}\left(\frac{\ln 2}{2} + \frac{\pi}{4}\right)}$

11. Alice and Bob are playing a game with n stones. They alternate turns with Alice starting, and each turn, someone takes a number of stones from the pile that is a divisor of the total number of stones. The person to take the last stone loses. For a pile of n stones, let $S(n)$ be the set of all possible numbers of stones that Alice may take in her first move so that she wins with perfect play, and let $A(n)$ be the sum of all numbers in $S(n)$. Estimate, as a positive integer, the value of

$$\sum_{n=1}^{4000} A(n).$$

If your estimate is E and the correct answer A, you will receive $\max\left(0, 3 - 2\max\left(\frac{A}{E}, \frac{E}{A}\right)\right)$ points for a valid estimate, and 0 points for an invalid estimate.

Answer: 1644849

Solution: Let $N = 4000$. First, we must realize that Alice wins when n is even and loses when n is odd. We can see this by induction: $n = 1$ is clearly losing, and from any odd number we can only move to even numbers, as all of the divisors of an odd number are odd. So, our strategy is that on every even number we can move to an odd number by taking 1 stone, and the opponent will be forced to move back to a smaller even number, and by induction this is winning for us.

Thus, the value of $A(n)$ will be the sum of the odd divisors of n for an even number, and 0 for an odd number. We now need to sum the odd divisors of even integers at most N. We can do this by counting the number of times each odd number $2k-1$ appears as a divisor of an even number at most N, which is $\left\lfloor \frac{N/2}{2k-1} \right\rfloor$. Then, summing over all odd numbers less than $N/2$ to get the total sum of all odd divisors gives

$$\sum_{k=1}^{N/4}(2k-1)\left\lfloor \frac{N/2}{2k-1} \right\rfloor.$$

If we estimate this as roughly $\frac{N}{4} \cdot \frac{N}{2} = \frac{N^2}{8}$, we will get an answer of $2 \cdot 10^6$, which is off by about $3.5 \cdot 10^5$ or roughly 20%, scoring about 0.568 points.

We can approximate this sum better by breaking into cases on the value of $l := \left\lfloor \frac{N/2}{2k+1} \right\rfloor$. For fixed l, the average value of $(2k+1) \cdot \left\lfloor \frac{N/2}{2k+1} \right\rfloor$ over all values of k corresponding to this value of l would be approximately $\frac{N}{2} \cdot \frac{2l+1}{2l+2}$, with approximately $\frac{N}{4} \cdot \frac{1}{l(l+1)}$ many such k. We get the following sum.

$$\frac{N^2}{8} \sum_{l=1}^{N/2} \left(1 - \frac{1}{2l+2}\right) \cdot \left(\frac{1}{l} - \frac{1}{l+1}\right)$$

This sum can be simplified as follows:

$$\begin{aligned}
\frac{N^2}{8} \sum_{l=1}^{N/2} \left(1 - \frac{1}{2l+2}\right) \cdot \left(\frac{1}{l} - \frac{1}{l+1}\right) &= \frac{N^2}{8} \sum_{l=1}^{N/2} \left(\frac{1}{l(l+1)} - \frac{1}{l(2l+2)} + \frac{1}{(l+1)(2l+2)}\right) \\
&= \frac{N^2}{16} \sum_{l=1}^{N/2} \left(\frac{1}{l(l+1)} + \frac{1}{(l+1)^2}\right) \\
&= \frac{N^2}{16} \left(\sum_{l=1}^{N/2} \left(\frac{1}{l} - \frac{1}{l+1}\right) + \sum_{l=1}^{N/2} \frac{1}{(l+1)^2}\right) \\
&= \frac{N^2}{16} \left(1 - \frac{1}{N/2+1} + \sum_{l=1}^{N/2+1} \frac{1}{l^2} - 1\right) \\
&\approx \frac{N^2}{16} \left(\frac{\pi^2}{6} - \frac{1}{N/2+1}\right) = \frac{N^2}{8} \left(\frac{\pi^2}{12} - \frac{1}{N+2}\right)
\end{aligned}$$

Plugging in $N = 4000$ gives $2 \cdot 10^6 \cdot (\frac{\pi^2}{12} - \frac{1}{N+2})$ or $10^6 \cdot (\frac{\pi^2}{6} - \frac{2}{N+2})$. Using the value of 1.645 for $\frac{\pi^2}{6}$ gives an estimate of roughly 1644500, which is 349 away from the true answer (although it gets slightly worse with better estimations of $\frac{\pi^2}{6}$). If we use an integral bound to get that $\sum_{k=N+1}^{\infty} \frac{1}{k^2} \in [1/(N+1), 1/N]$ we essentially cancel the $\frac{1}{N}$ term, which refines this estimation to about 1645000. With all the aforementioned methods and a perfect estimate of $\frac{\pi^2}{6}$ the obtained estimate is about 1644934, with an absolute error of 85. All of these estimates receive essentially a perfect score (> 0.999).

12. A point P is selected uniformly at random from the surface of a sphere centered at $O = (2, 0, 0)$ with radius 2 (so that a point is equally likely to be selected from two regions of equal area on the surface). The segment ℓ is drawn with P as an endpoint and O as its midpoint. Let $S(\ell)$ be the surface created by revolving ℓ about the y-axis, let M be the maximum y-coordinate of any point in ℓ, and let $V(\ell)$ be the volume of the solid enclosed by $S(\ell)$ and the planes $y = \pm M$. Find the expected value of $V(\ell)$.

Answer: $\frac{28\pi}{3}$

Solution: We first solve for a fixed rotation, and then average the value over all possible rotations. The segment ℓ can be determined by the value of $M \in [0, 2]$ and its rotation around the line $x = 2, z = 0$ by a single parameter $\theta \in [0, 2\pi)$. Without loss of generality we can assume P is above the yz-plane, so M is the y-coordinate of P. Uniform distributions over M and θ give a uniform distribution over ℓ (as we are essentially choosing a random point from a hemisphere, and projections from a uniform distribution on a sphere onto a diameter are uniform on the diameter. This can be shown by considering an integral over the surface area of a sphere in cylindrical coordinates).

Thus, we may denote a segment ℓ by its midpoint $\mathbf{m} = (2, 0, 0)$ and a length 2 direction vector $\mathbf{u} = \left(\sqrt{4 - M^2}\cos\theta, M, \sqrt{4 - M^2}\sin\theta\right)$. To compute $V(\ell)$ for this segment, we have to integrate

$$\int_{-M}^{M} \pi(R(t))^2 \, dt,$$

where $R(t)$ is a function of the distance of the point on l with y-coordinate t from the y-axis. Since l is a segment, we can compute this as a function of t only: we use the unit direction vector to compute what the x and z coordinates must be of this point, multiplying the x and z components by $\frac{t}{M}$. This gives

$$\begin{aligned} R(t)^2 &= \left(2 + \frac{t}{M}\sqrt{4 - M^2}\cos\theta\right)^2 + \left(\frac{t}{M}\sqrt{4 - M^2}\sin\theta\right)^2 \\ &= 4 + \frac{4t}{M}\sqrt{4 - M^2}\cos\theta + \frac{t^2}{M^2}(4 - M^2)\cos^2\theta + \frac{t^2}{M^2}(4 - M^2)\sin^2\theta \\ &= 4 + \frac{4t}{M}\sqrt{4 - M^2}\cos\theta + \frac{t^2}{M^2}(4 - M^2) \end{aligned}$$

Factoring out the constant π, we have

$$V(\ell) = \pi \int_{-M}^{M} \left(4 + \frac{4t}{M}\sqrt{4 - M^2}\cos\theta + \frac{t^2}{M^2}(4 - M^2)\right) dt$$

We note that the second term is an odd function of t, so it will cancel over this integral, leaving us with

$$\begin{aligned} V(\ell) &= \pi \int_{-M}^{M} \left(4 + \frac{t^2}{M^2}(4 - M^2)\right) dt \\ &= \pi \left(4t + \frac{t^3}{3M^2}(4 - M^2)\right)\Bigg|_{-M}^{M} \\ &= \pi \left(8M + \frac{2M}{3}\left(4 - M^2\right)\right) \end{aligned}$$

Note that this (surprisingly) doesn't depend on θ, so we can take the expected value with respect to M only, which leaves us with one last integral:

$$\begin{aligned}\mathbb{E}[V(\ell)] &= \frac{1}{2}\int_0^2 \pi\left(8M + \frac{2M}{3}\left(4 - M^2\right)\right) dM \\ &= \pi\int_0^2 \left(4M + \frac{4M}{3} - \frac{M^3}{3}\right) dM \\ &= \pi\left(8 + \frac{8}{3} - \frac{16}{12}\right) \\ &= \boxed{\frac{28\pi}{3}}\end{aligned}$$

Alternate solution: Using spherical coordinates, and parameterizing the angle of ℓ by (φ, θ), with $\varphi \in [0, \pi)$ and $\theta \in [0, \pi)$, we have that

$$V(\theta, \varphi) := V(\ell) = \int_{-2}^{2} \pi((2 + r\sin\theta\cos\varphi)^2 + (r\cos\theta)^2)\sin\theta\sin\varphi\,\mathrm{d}r\,,$$

which can be computed to be

$$16\pi\left(1 + \frac{1}{3}\left(\sin^2\theta\cos^2\varphi + \cos^2\theta\right)\right)\sin\theta\sin\varphi.$$

We now need to compute the average over (φ, θ) chosen uniformly at random from the surface of the cube. The probability measure in this case is

$$\frac{4\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi}{8\pi} = \frac{2}{\pi}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\,,$$

and hence the expected value can be computed to be

$$\frac{2}{\pi}\int_0^\pi\int_0^\pi V(\theta, \varphi)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = \boxed{\frac{28\pi}{3}}.$$