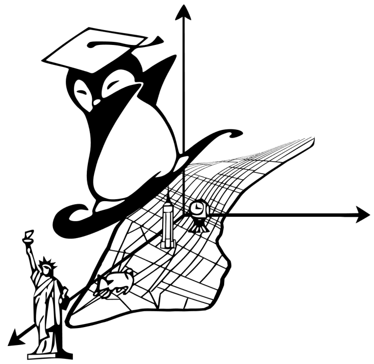
ICMT — Individual Round (Division A)



Do not flip open this packet until instructed by your proctor.

Round Structure

- This is an **individual** test. You will have **75 minutes** total.
- There are **12 problems** total. Problems 5 and 6 are estimation problems; you will get more points for an answer *closer* to the exact answer. The remaining 10 problems are standard short answer problems.
- Problems are roughly ordered by difficulty.

Scoring

- The 10 standard short answer problems are worth 1 point for a correct answer, and 0 points for an incorrect answer.
- The 2 estimation problems have separate scoring systems, which will be explained in their respective problem statements. Each estimation problem will give at least 0 points and at most 1 point.
- Your total score on the round will be the sum of the points scored on all 12 problems.
- In case of a tie, we will implement the following tiebreaker system:
  1. First, the 10 short answer problems are ordered from hardest to easiest, based on the number of correct submissions.
     - The estimation problems will not be taken into account for the tiebreaker system.
     - In case of short answer problems having an equal number of solves, the problems will then be ordered from later in the test to earlier.
  2. The n^{th} hardest problem is assigned a tiebreaking value of 2^{-n}. For example, the hardest problem will have a tiebreaking value of $\frac{1}{2}$, the second hardest problem will have a tiebreaking value of $\frac{1}{4}$, et cetera.
  3. Each student's tiebreaking index is calculated as the sum of their original score and the tiebreaker values of the problems that they correctly answered.
  4. Students are ordered by their tiebreaking index, determining their tiebroken rank.

  Roughly speaking, among those tied for the same score, whoever solved the hardest short answer problem is placed the highest, followed by the one who solved the next hardest, and so on.

1. Pick a set $S \subseteq \{1, 2, \ldots, 20\}$ such that no two (not necessarily distinct) numbers in S have a sum that is a multiple of 5. In other words, for any $a, b \in S$ (where a and b need not be distinct), we require $a + b \not\equiv 0 \pmod 5$. What is the largest possible size of such a set S?

2. Compute the smallest integer C such that $a + b + c + 20d + 26e < C$ for all real numbers a, b, c, d, e where $a^2 + b^2 + c^2 + d^2 + e^2 \leq 1$.

3. Suppose $y(x)$ satisfies the differential equation $(x^2 + 1)\frac{dy}{dx} + 2xy = x$ with $y(0) = 0$. Compute

$$\int_0^1 (x^2 + 1)\, y(x)\, dx.$$

4. Two gamblers, Nikhil and Aditya, each start with \$1. In each round, independently for each gambler, their fortune either increases by \$1 with probability $\frac{2}{3}$ or decreases by \$1 with probability $\frac{1}{3}$. All rounds and gamblers' outcomes are mutually independent. The game ends as soon as either gambler's fortune reaches \$0. What is the probability that the game continues forever?

5. 1000 points are placed independently and uniformly at random on the unit sphere in $\mathbb{R}^3$ centered at O. Two points P, Q are called *close* if $m\angle POQ \leq 5^\circ$. Estimate, as an integer or a decimal rounded to *at most two decimal places*, the expected number of close unordered pairs among the 1000 points. If your estimate is E and the correct answer is A, you will receive $\exp\left(-4\frac{|E-A|}{A}\right)$ points for a valid estimate and 0 points for an invalid estimate.

6. Define $d(n)$ as the number of divisors of n and $\nu_2(m)$ as the maximum integer k such that 2^k divides m. Estimate, as a positive integer, the value of

$$\sum_{i=1}^{2026} \nu_2(d(i)).$$

If your estimate is E and the correct answer is A, you will receive $\left(\min\left(\frac{E}{A}, \frac{A}{E}\right)\right)^3$ points for a valid estimate and 0 points for an invalid estimate.

7. Fix a positive integer N. Each permutation $\sigma \in S_5$ defines a bijection on $(\mathbb{Z}/N\mathbb{Z})^5$ by permuting coordinates:

$$(x_1, \ldots, x_5) \mapsto (x_{\sigma(1)}, \ldots, x_{\sigma(5)}).$$

This bijection induces a permutation $\pi_{\sigma,N}$ of the N^5 elements of $(\mathbb{Z}/N\mathbb{Z})^5$. Let $\varepsilon_N(\sigma) = \operatorname{sgn}(\pi_{\sigma,N})$. Determine the number of $N \leq 100$ for which there exists $\sigma \in S_5$ such that $\varepsilon_N(\sigma) = -1$.

8. Compute the coefficient of the $(x_1 x_2 \cdots x_{2026})^{2025}$ term in the multivariate polynomial

$$\prod_{1 \leq i < j \leq 2026} (x_i - x_j)^2.$$

9. Compute the value of

$$\int_{-\infty}^{\infty} \frac{x \sin(\pi x)}{x^2 + 2x + 4}\, dx.$$

10. Define $\mathbb{S}^{2\times 2}$ as the set of symmetric real 2×2 matrices and $\mathbf{0}$ the zero matrix in $\mathbb{S}^{2\times 2}$. Consider functions $f : \mathbb{Z} \to \mathbb{S}^{2\times 2}$ satisfying the following properties:

- $f(mn) = f(m)f(n)$ for all integers m and n,
- $f(n) = f(n + 27000)$ for all integers n,
- $f(n) = \mathbf{0}$ if $\gcd(27000, n) > 1$.

Find the size of the largest set S of such functions where for any two distinct functions $f, g \in S$, there does *not* exist an invertible matrix A such that $f(x) = Ag(x)A^{-1}$ for all $x \in \mathbb{Z}$.

11. Let $\{t\}$ denote the fractional part of a real number t, and define $s(t) = \{t\} - \frac{1}{2}$. For a function $f : (0, 1) \to \mathbb{R}$, suppose the limits

$$A = \lim_{x \to 0^+} x f(x), \qquad B = \lim_{x \to 1^-} (1 - x) f(x)$$

exist and are finite. The Hadamard finite-part integral of f over $(0, 1)$ is defined by

$$\mathrm{FP} \int_0^1 f(x)\, dx = \lim_{\varepsilon \to 0^+} \left(\int_\varepsilon^{1-\varepsilon} f(x)\, dx - (A + B) \log \frac{1}{\varepsilon} \right),$$

provided the limit exists. Evaluate

$$\lim_{N \to \infty} \frac{1}{N \log N} \sum_{n=1}^{N} \mathrm{FP} \int_0^1 \frac{s(nx)\, s(n(1-x))}{x(1-x)}\, dx.$$

12. For any commutative ring R, let $\mathrm{GL}_n(R)$ denote the (multiplicative) group of invertible $n \times n$ matrices with entries from R. Compute the maximum possible order of an element in $\mathrm{GL}_{12}(\mathbb{Z}/26\mathbb{Z})$.