

ICMT — Individual Round Solutions (Division A)

1. Pick a set $S \subseteq \{1, 2, \dots, 20\}$ such that no two (not necessarily distinct) numbers in S have a sum that is a multiple of 5. In other words, for any $a, b \in S$ (where a and b need not be distinct), we require $a + b \not\equiv 0 \pmod{5}$. What is the largest possible size of such a set S ?

Answer: 8

Solution: Partition $\{1, \dots, 20\}$ into residue classes mod 5. Each class \bar{r} has 4 elements.

The condition $a + b \not\equiv 0 \pmod{5}$ means no residue r may appear together with its complement $5 - r$, and any residue r with $2r \equiv 0 \pmod{5}$ cannot appear at all. Thus $\bar{0}$ is forbidden, and from each complementary pair $\{\bar{1}, \bar{4}\}, \{\bar{2}, \bar{3}\}$, we may choose at most one class.

Each allowed class has size 4, so the largest possible size of S is $4 + 4 = \boxed{8}$. One optimal choice is $S = \{1, 6, 11, 16\} \cup \{2, 7, 12, 17\}$.

2. Compute the smallest integer C such that $a + b + c + 20d + 26e < C$ for all real numbers a, b, c, d, e where $a^2 + b^2 + c^2 + d^2 + e^2 \leq 1$.

Answer: 33

Solution: Consider the linear expression $L = a + b + c + 20d + 26e$. This is maximized when the vector (a, b, c, d, e) is in the direction of the vector $\mathbf{v} = (1, 1, 1, 20, 26)$.

Cauchy-Schwarz implies that $L \leq (\sqrt{a^2 + b^2 + c^2 + d^2 + e^2})(\sqrt{1^2 + 1^2 + 1^2 + 20^2 + 26^2})$.

Since $a^2 + b^2 + c^2 + d^2 + e^2 \leq 1$, we have $L \leq \sqrt{1^2 + 1^2 + 1^2 + 20^2 + 26^2} = \sqrt{1079}$.

Since $32^2 = 1024$ and $33^2 = 1089$, the smallest integer C satisfying the condition is $\boxed{33}$.

3. Suppose $y(x)$ satisfies the differential equation $(x^2 + 1)\frac{dy}{dx} + 2xy = x$ with $y(0) = 0$. Compute

$$\int_0^1 (x^2 + 1)y(x) dx.$$

Answer: $\frac{1}{6}$

Solution: Notice that the left-hand side is the derivative of $(x^2 + 1)y$, since $\frac{d}{dx}[(x^2 + 1)y] = (x^2 + 1)y' + 2xy$.

Therefore, the ODE becomes $\frac{d}{dx}[(x^2 + 1)y] = x$, and integrating from 0 to x gives

$$(x^2 + 1)y = \int_0^x t dt = x^2/2,$$

so $y(x) = x^2/[2(x^2 + 1)]$.

Then the desired integral is

$$\int_0^1 (x^2 + 1)y(x) dx = \int_0^1 (x^2 + 1) \cdot \frac{x^2}{2(x^2 + 1)} dx = \frac{1}{2} \int_0^1 x^2 dx = \frac{1}{2} \cdot \frac{1}{3} = \boxed{\frac{1}{6}}.$$

4. Two gamblers, Nikhil and Aditya, each start with \$1. In each round, independently for each gambler, their fortune either increases by \$1 with probability $\frac{2}{3}$ or decreases by \$1 with probability $\frac{1}{3}$. All rounds and gamblers' outcomes are mutually independent. The game ends as soon as either gambler's fortune reaches \$0. What is the probability that the game continues forever?

Answer: $\frac{1}{4}$

Solution: Let $p = \frac{2}{3}$ and $q = \frac{1}{3}$. For a single gambler starting at $i \geq 1$, let r_i be the probability of ever reaching 0. Then

$$r_i = q r_{i-1} + p r_{i+1}, \quad r_0 = 1,$$

whose bounded solution is

$$r_i = \left(\frac{q}{p}\right)^i.$$

Thus $r_1 = \frac{q}{p} = \frac{1}{2}$, so the probability a single gambler never reaches 0 is $1 - \frac{1}{2} = \frac{1}{2}$. Since the two gamblers act independently, the probability that both avoid 0 forever is $\left(\frac{1}{2}\right)^2 = \boxed{\frac{1}{4}}$.

5. 1000 points are placed independently and uniformly at random on the unit sphere in \mathbb{R}^3 centered at O . Two points P, Q are called *close* if $m\angle POQ \leq 5^\circ$. Estimate, as an integer or a decimal rounded to *at most two decimal places*, the expected number of close unordered pairs among the 1000 points. If your estimate is E and the correct answer is A , you will receive $\exp\left(-4\frac{|E-A|}{A}\right)$ points for a valid estimate and 0 points for an invalid estimate.

Answer: 950.37

Solution: There are $\binom{1000}{2} = 499500$ unordered pairs. For a fixed point, the fraction of the sphere lying within angular radius θ is the ratio of a spherical cap area to the total sphere area:

$$\frac{2\pi(1 - \cos \theta)}{4\pi} = \frac{1 - \cos \theta}{2}.$$

With $\theta = 5^\circ$, our answer by linearity of expectation is $499500\left(\frac{1 - \cos 5^\circ}{2}\right) \approx \boxed{950.37}$.

A better approximation, for scoring purposes is 950.3741515865533.

6. Define $d(n)$ as the number of divisors of n and $\nu_2(m)$ as the maximum integer k such that 2^k divides m . Estimate, as a positive integer, the value of

$$\sum_{i=1}^{2026} \nu_2(d(i)).$$

If your estimate is E and the correct answer is A , you will receive $\left(\min\left(\frac{E}{A}, \frac{A}{E}\right)\right)^3$ points for a valid estimate and 0 points for an invalid estimate.

Answer: 4071

Solution: For each prime at most n , note that it contributes exactly $\lfloor \frac{n}{p} \rfloor + \lfloor \frac{n}{p^3} \rfloor + \dots$ to the total $\nu_2(\cdot)$. This gives us

$$\sum_{i=1}^n \nu_2(d(n)) \leq n \cdot \sum_{p \leq n} \frac{p}{p^2 - 1} \sim n \cdot \sum_{p \leq n} \frac{1}{p} \sim n \ln \ln n$$

and for the lower bound

$$\sum_{i=1}^n \nu_2(d(n)) \geq n \cdot \sum_{p \leq n} \frac{p}{p^2 - 1} - n \ln n \cdot \sum_{p \leq n} 1$$

and the latter sum is asymptotic to n .

We estimate

$$S(N) = \sum_{n=1}^N \nu_2(d(n))$$

using the asymptotic main term $N \ln \ln N$.

For $N = 2026$,

$$\ln 2026 \approx 7.613, \quad \ln(\ln 2026) \approx \ln(7.613) \approx 2.030.$$

Therefore a reasonable estimate is

$$N \ln \ln N \approx 2026 \times 2.030 \approx 4113.$$

The exact value (computed directly) is $S(2026) = \boxed{4071}$.

7. Fix a positive integer N . Each permutation $\sigma \in S_5$ defines a bijection on $(\mathbb{Z}/N\mathbb{Z})^5$ by permuting coordinates:

$$(x_1, \dots, x_5) \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(5)}).$$

This bijection induces a permutation $\pi_{\sigma, N}$ of the N^5 elements of $(\mathbb{Z}/N\mathbb{Z})^5$. Let $\varepsilon_N(\sigma) = \text{sgn}(\pi_{\sigma, N})$. Determine the number of $N \leq 100$ for which there exists $\sigma \in S_5$ such that $\varepsilon_N(\sigma) = -1$.

Answer: 25

Solution: The main claim is that

$$\varepsilon_N \stackrel{?}{=} \begin{cases} 1 & \text{if } N \equiv 0, 1, 2 \pmod{4}, \\ \text{sgn} & \text{if } N \equiv 3 \pmod{4}, \end{cases}$$

where 1 is the trivial character, and sgn is the sign character; the answer is then quickly seen to be $\lfloor \frac{100+1}{4} \rfloor = \boxed{25}$.

It remains to show the main claim. Because S_5 is generated by transpositions, it is enough to compute ε_N on transpositions τ . Up to relabeling, we may assume that $\tau = (1, 2)$. It is now enough to show that

$$\varepsilon_N(\tau) \stackrel{?}{=} \begin{cases} +1 & \text{if } N \equiv 0, 1, 2 \pmod{4}, \\ -1 & \text{if } N \equiv 3 \pmod{4}. \end{cases}$$

We will do this by an explicit calculation by decomposing the action of τ into $(\mathbb{Z}/N\mathbb{Z})^{\oplus 5}$ into cycles. Namely, given a 5-tuple $a = (a_1, a_2, a_3, a_4, a_5)$, we see that the action of τ fixes a if and only if $a_1 = a_2$. Thus, the action of τ decomposes into fixed points and orbits of size 2, and the number of orbits of size 2 equals the number of orbits of tuples a with $a_1 \neq a_2$. We see that the number of such orbits is $N(N-1)/2 \cdot N^3$, so we may also say that the action of τ on $(\mathbb{Z}/N\mathbb{Z})^{\oplus 5}$ is given by $N(N-1)/2 \cdot N^3$ transpositions, so the sign is

$$\varepsilon_N(\tau) = (-1)^{N(N-1)/2 \cdot N^3}.$$

The right-hand side now only depends on $N(N-1)/2 \cdot N^3 \pmod{2}$ and so depends only on $N \pmod{4}$, so the main claim follows by some straightforward casework.

8. Compute the coefficient of the $(x_1 x_2 \cdots x_{2026})^{2025}$ term in the multivariate polynomial

$$\prod_{1 \leq i < j \leq 2026} (x_i - x_j)^2.$$

Answer: $-2026!$

Solution: Let $k = 2026$. Note that if we define the Vandermonde matrix $V(x_1, \dots, x_k) = (x_i^{j-1})_{1 \leq i, j \leq k}$, then $\det(V) = \prod_{1 \leq i < j \leq k} (x_j - x_i)$. On the other hand, $\det(V) = \sum_{\sigma \in S_k} \text{sgn}(\sigma) \prod_{i=1}^k x_i^{\sigma(i)-1}$ by the formula for determinants. Thus,

$$\prod_{1 \leq i < j \leq k} (x_i - x_j)^2 = \det(V)^2 = \sum_{\sigma, \tau \in S_k} \text{sgn}(\sigma) \text{sgn}(\tau) \prod_{i=1}^k x_i^{\sigma(i)+\tau(i)-2}$$

The coefficient of $(x_1 \cdots x_k)^{k-1}$ in the above polynomial is $\sum_{\sigma, \tau \in S_k: \sigma(i)+\tau(i)=k+1} \text{sgn}(\sigma) \text{sgn}(\tau)$. Now, define the permutation $\pi: [k] \rightarrow [k]$ as $\pi(i) = k+1-i$. Then in the above summation, we have $\tau = \pi\sigma$. Since $S_k \ni \sigma \mapsto \text{sgn}(\sigma) \in \{\pm 1\}$ is a group homomorphism, we have $\text{sgn}(\sigma) \text{sgn}(\tau) = \text{sgn}(\sigma) \text{sgn}(\pi\sigma) = \text{sgn}(\sigma) \text{sgn}(\pi) \text{sgn}(\sigma) = \text{sgn}(\sigma)^2 \text{sgn}(\pi) = \text{sgn}(\pi)$. Consequently, the coefficient is $\text{sgn}(\pi) \cdot k!$, and thus $C = (-1)^{\lfloor k/2 \rfloor} k! = \boxed{-2026!}$.

9. Compute the value of

$$\int_{-\infty}^{\infty} \frac{x \sin(\pi x)}{x^2 + 2x + 4} dx.$$

Answer: $-\pi e^{-\pi\sqrt{3}}$

Solution: Since $\sin(\pi x) = \text{Im}(e^{i\pi x})$, we write

$$\int_{-\infty}^{\infty} \frac{x \sin(\pi x)}{x^2 + 2x + 4} dx = \text{Im} \int_{-\infty}^{\infty} \frac{x e^{i\pi x}}{x^2 + 2x + 4} dx.$$

Define

$$f(z) = \frac{z e^{i\pi z}}{z^2 + 2z + 4}.$$

The denominator satisfies

$$z^2 + 2z + 4 = (z - (-1 + i\sqrt{3}))(z - (-1 - i\sqrt{3})),$$

so f has simple poles at $z = -1 \pm i\sqrt{3}$, and only $z_0 = -1 + i\sqrt{3}$ lies in the upper half-plane. Let C_R be the semicircular contour consisting of $[-R, R]$ together with the upper semicircle Γ_R . On Γ_R , we have $|e^{i\pi z}| = e^{-\pi \text{Im } z} \leq 1$, and for large R ,

$$|z^2 + 2z + 4| \geq R^2 - (2R + 4) \geq \frac{1}{2}R^2.$$

Thus $|f(z)| \leq 2/R$ on Γ_R . Splitting the arc into a central portion (where $e^{-\pi R \sin t}$ gives exponential decay) and small end pieces of arbitrarily small length, one concludes that $\int_{\Gamma_R} f(z) dz \rightarrow 0$ as $R \rightarrow \infty$.

Hence, by the residue theorem,

$$\int_{-\infty}^{\infty} \frac{x e^{i\pi x}}{x^2 + 2x + 4} dx = 2\pi i \text{Res}(f; z_0).$$

Since the pole is simple,

$$\text{Res}(f; z_0) = \frac{z_0 e^{i\pi z_0}}{2z_0 + 2}.$$

Now $2z_0 + 2 = 2i\sqrt{3}$, and

$$e^{i\pi z_0} = e^{i\pi(-1+i\sqrt{3})} = -e^{-\pi\sqrt{3}}.$$

Therefore

$$\text{Res}(f; z_0) = \frac{(1 - i\sqrt{3})e^{-\pi\sqrt{3}}}{2i\sqrt{3}}.$$

Multiplying by $2\pi i$ gives

$$\int_{-\infty}^{\infty} \frac{x e^{i\pi x}}{x^2 + 2x + 4} dx = \left(\frac{\pi}{\sqrt{3}} - i\pi \right) e^{-\pi\sqrt{3}}.$$

Taking the imaginary part, our answer is

$$\int_{-\infty}^{\infty} \frac{x \sin(\pi x)}{x^2 + 2x + 4} dx = \boxed{-\pi e^{-\pi\sqrt{3}}}.$$

10. Define $\mathbb{S}^{2 \times 2}$ as the set of symmetric real 2×2 matrices and $\mathbf{0}$ the zero matrix in $\mathbb{S}^{2 \times 2}$. Consider functions $f : \mathbb{Z} \rightarrow \mathbb{S}^{2 \times 2}$ satisfying the following properties:

- $f(mn) = f(m)f(n)$ for all integers m and n ,
- $f(n) = f(n + 27000)$ for all integers n ,
- $f(n) = \mathbf{0}$ if $\text{gcd}(27000, n) > 1$.

Find the size of the largest set S of such functions where for any two distinct functions $f, g \in S$, there does *not* exist an invertible matrix A such that $f(x) = Ag(x)A^{-1}$ for all $x \in \mathbb{Z}$.

Answer: 153

Solution: Let $N = 2$ be the dimension of our matrices and $K = 27000$. We begin by noting some facts about the matrices in the range of a specific function f . First, we know that real symmetric matrices have all real eigenvalues (which can be reproved by taking conjugates of $v^t Av$ for some eigenvector v with eigenvalue λ). Since $f(1) = f(1 \cdot 1) = f(1)^2$, the only eigenvalues of $f(1)$ are 0 and 1. For all n coprime to 27000, we have that $f(n)^k = f(n^k) = f(1)$ for some k , implying that the only eigenvalues we can have are $0, \pm 1$.

We also know that all outputs of f commute with each other, since $f(m)f(n) = f(mn) = f(nm) = f(n)f(m)$. In addition to the fact that real symmetric matrices are all diagonalizable, this tells us that all outputs can be diagonalized by the same invertible matrix A : in particular, we can simply change our basis and consider only diagonal matrices with eigenvalues of $0, \pm 1$.

You may not be convinced that commuting matrices that are diagonalizable can all be diagonalized by the same matrix A . At a high level, commuting matrices preserve each other's eigenspaces (as their transformations must also commute, which can only happen if they're scaled in the same directions). Diagonalization involves transforming the coordinates to show the actions of a linear transformation on its eigenspaces. However, the diagonalization matrices only depend on what the eigenspaces are, not the actual scaling factor on each eigenspace. Therefore, since commuting matrices preserve eigenspaces, they must act by simply scaling all axes of the diagonalization matrix, which means their matrix in the new coordinate system must also be diagonal.

Now, we are essentially considering functions from \mathbb{Z} to real diagonal matrices with $0, \pm 1$ on the diagonal. Each diagonal entry is a function $\chi : \mathbb{Z} \rightarrow \{-1, 0, 1\}$ satisfying the following conditions:

- $\chi(mn) = \chi(m)\chi(n)$ for all m, n ,
- $\chi(n) = \chi(n + K)$ for all n ,
- $\chi(n) = 0$ if $\gcd(K, n) > 1$.

Suppose the number of such functions is $k = k(K)$. Since two matrices are similar iff they have the same multiset of eigenvalues, each multiset of such functions χ of size N will give a unique element in S , thus the answer is

$$\binom{k + N - 1}{N} = \binom{k + 1}{2} = \frac{(k + 1)k}{2}.$$

Now we need to find k . Suppose $\chi(1) = 0$, then for any n coprime with K , we have $n^r \equiv 1 \pmod{K}$ for some r , and thus $0 = \chi(1) = \chi(n^r) = \chi(n)^r$ and so $\chi(n) = 0$. Similarly, if $\chi(1) = 1$ then $\chi(1) = \chi(n^r) = \chi(n)^r$ so $\chi(n) \neq 0$. Therefore, there are two cases: either $\chi \equiv 0$, or $\chi(n) \neq 0$ for all n coprime with K , in which case it reduces to a homomorphism from $(\mathbb{Z}/K\mathbb{Z})^* \rightarrow \{\pm 1\} = \mathbb{Z}/2\mathbb{Z}$. Hence,

$$k(K) = 1 + |\text{hom}((\mathbb{Z}/K\mathbb{Z})^*, \mathbb{Z}/2\mathbb{Z})|.$$

Since $K = 27000 = 2^3 \cdot 3^3 \cdot 5^3$, we have

$$\mathbb{Z}/K\mathbb{Z} \cong C_{2^3} \times C_{3^3} \times C_{5^3},$$

so

$$\begin{aligned} (\mathbb{Z}/K\mathbb{Z})^* &\cong C_{2^3}^* \times C_{3^3}^* \times C_{5^3}^* \cong C_2 \times C_2 \times C_{2 \cdot 3^2} \times C_{4 \cdot 5^2} \\ &\cong C_2^3 \times C_4 \times C_{3^2} \times C_{5^2}, \end{aligned}$$

so

$$\begin{aligned} \text{hom}((\mathbb{Z}/K\mathbb{Z})^*, \mathbb{Z}/2\mathbb{Z}) &= \text{hom}(C_2^3, \mathbb{Z}/2\mathbb{Z}) \times \text{hom}(C_4, \mathbb{Z}/2\mathbb{Z}) \times \text{hom}(C_{3^2}, \mathbb{Z}/2\mathbb{Z}) \times \text{hom}(C_{5^2}, \mathbb{Z}/2\mathbb{Z}) \\ &= C_2^3 \times C_2 \\ &= C_2^4. \end{aligned}$$

Therefore,

$$k(K) = 1 + |\text{hom}((\mathbb{Z}/K\mathbb{Z})^*, \mathbb{Z}/2\mathbb{Z})| = 1 + |C_2^4| = 1 + 16 = 17,$$

hence the answer is

$$\frac{(k+1)k}{2} = \frac{18 \cdot 17}{2} = \boxed{153}.$$

11. Let $\{t\}$ denote the fractional part of a real number t , and define $s(t) = \{t\} - \frac{1}{2}$. For a function $f : (0, 1) \rightarrow \mathbb{R}$, suppose the limits

$$A = \lim_{x \rightarrow 0^+} xf(x), \quad B = \lim_{x \rightarrow 1^-} (1-x)f(x)$$

exist and are finite. The Hadamard finite-part integral of f over $(0, 1)$ is defined by

$$\text{FP} \int_0^1 f(x) dx = \lim_{\varepsilon \rightarrow 0^+} \left(\int_\varepsilon^{1-\varepsilon} f(x) dx - (A+B) \log \frac{1}{\varepsilon} \right),$$

provided the limit exists. Evaluate

$$\lim_{N \rightarrow \infty} \frac{1}{N \log N} \sum_{n=1}^N \text{FP} \int_0^1 \frac{s(nx) s(n(1-x))}{x(1-x)} dx.$$

Answer: $\frac{1}{3}$

Solution: Let $\{t\} = t - [t]$ and $s(t) = \{t\} - \frac{1}{2}$. For a.e. $x \in (0, 1)$ with $nx \notin \mathbb{Z}$ we have $\{n(1-x)\} = \{n-nx\} = \{-nx\} = 1 - \{nx\}$, hence $s(n(1-x)) = -s(nx)$ and therefore

$$\text{FP} \int_0^1 \frac{s(nx) s(n(1-x))}{x(1-x)} dx = - \text{FP} \int_0^1 \frac{s(nx)^2}{x(1-x)} dx = -K_n.$$

Partition $[0, 1)$ into $I_k = [k/n, (k+1)/n)$, where $x = (k+u)/n$ with $u \in [0, 1)$. Then $\{nx\} = u$, $s(nx) = u - \frac{1}{2}$, $dx = du/n$, and $x(1-x) = (k+u)(n-k-u)/n^2$, so for the truncated integral (removing $(0, \varepsilon)$ and $(1-\varepsilon, 1)$) one gets

$$\int_\varepsilon^{1-\varepsilon} \frac{s(nx)^2}{x(1-x)} dx = n \sum_{k=0}^{n-1} \int_0^1 \frac{(u - \frac{1}{2})^2}{(k+u)(n-k-u)} du + O(1),$$

where the $O(1)$ accounts for the boundary terms from $k = 0, n-1$ for $\varepsilon \asymp 1/n$. For $1 \leq k \leq n-1$ set

$J_{n,k} = \int_0^1 \frac{(u - \frac{1}{2})^2}{(k+u)(n-k-u)} du$ and let $J_{n,k} = \frac{1}{k(n-k)} \int_0^1 (u - \frac{1}{2})^2 du + E_{n,k} = \frac{1}{12k(n-k)} + E_{n,k}$, where

$$E_{n,k} = \int_0^1 (u - \frac{1}{2})^2 \left(\frac{1}{(k+u)(n-k-u)} - \frac{1}{k(n-k)} \right) du.$$

Since $\frac{1}{(k+u)(n-k-u)} - \frac{1}{k(n-k)} = \frac{2ku-nu+u^2}{k(k-n)(k+u)(k-n+u)}$, the numerator is $O(n)$ while $(k+u)(k-n+u) \asymp k(n-k)$ uniformly for $1 \leq k \leq n-1, 0 \leq u \leq 1$, so $|E_{n,k}| \leq C \frac{n}{k^2(n-k)^2}$ and hence $n \sum_{k=1}^{n-1} E_{n,k} = O(1)$ because $\sum_{k=1}^{n-1} \frac{1}{k^2(n-k)^2} = O(n^{-2})$. It follows that

$$\int_\varepsilon^{1-\varepsilon} \frac{s(nx)^2}{x(1-x)} dx = \frac{n}{12} \sum_{k=1}^{n-1} \frac{1}{k(n-k)} + O(1) = \frac{1}{6} H_{n-1} + O(1) = \frac{1}{6} \log n + O(1).$$

In terms of the finite part, we compute the endpoint coefficients

$$A = \lim_{x \rightarrow 0^+} x \frac{s(nx)^2}{x(1-x)}, \quad B = \lim_{x \rightarrow 1^-} (1-x) \frac{s(nx)^2}{x(1-x)}.$$

As $x \rightarrow 0^+$, we have $\{nx\} \rightarrow 0$, hence $s(nx) \rightarrow -\frac{1}{2}$ and $s(nx)^2 \rightarrow \frac{1}{4}$, while $1-x \rightarrow 1$. Therefore

$$A = \lim_{x \rightarrow 0^+} \frac{s(nx)^2}{1-x} = \frac{1}{4}.$$

Similarly, as $x \rightarrow 1^-$, we have $\{nx\} \rightarrow 0$, so again $s(nx)^2 \rightarrow \frac{1}{4}$ and $x \rightarrow 1$, giving

$$B = \lim_{x \rightarrow 1^-} \frac{s(nx)^2}{x} = \frac{1}{4}.$$

Thus $A + B = \frac{1}{2}$, and by the definition of the Hadamard finite part,

$$K_n = \text{FP} \int_0^1 \frac{s(nx)^2}{x(1-x)} dx = \left(\frac{1}{6} \log n + O(1) \right) - \frac{1}{2} \log n = -\frac{1}{3} \log n + O(1).$$

Summing yields $\sum_{n=1}^N \text{FP} \int_0^1 \frac{s(nx)s(n(1-x))}{x(1-x)} dx = \frac{1}{3} \sum_{n=1}^N \log n + O(N) = \frac{1}{3} \log(N!) + O(N)$. Since $\log(N!) = N \log N - N + O(\log N)$ we conclude

$$\lim_{N \rightarrow \infty} \frac{1}{N \log N} \sum_{n=1}^N \text{FP} \int_0^1 \frac{s(nx)s(n(1-x))}{x(1-x)} dx = \boxed{\frac{1}{3}}.$$

12. For any commutative ring R , let $GL_n(R)$ denote the (multiplicative) group of invertible $n \times n$ matrices with entries from R . Compute the maximum possible order of an element in $GL_{12}(\mathbb{Z}/26\mathbb{Z})$.

Answer: $3937(13^{12} - 1)$

Solution: Since $26 = 2 \cdot 13$ with $\gcd(2, 13) = 1$, the Chinese Remainder Theorem gives

$$\mathbb{Z}/26\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/13\mathbb{Z},$$

hence

$$GL_{12}(\mathbb{Z}/26\mathbb{Z}) \cong GL_{12}(\mathbb{F}_2) \times GL_{12}(\mathbb{F}_{13}).$$

An element (A, B) has order $\text{lcm}(|A|, |B|)$, so we must maximize

$$\text{lcm}(|A|, |B|) = |B| \cdot \frac{|A|}{\gcd(|A|, |B|)}.$$

Over \mathbb{F}_q , the order $q^{12} - 1$ is attainable in $GL_{12}(\mathbb{F}_q)$. Indeed, if A has irreducible characteristic polynomial of degree 12, then A generates a copy of $\mathbb{F}_{q^{12}}$ inside $M_{12}(\mathbb{F}_q)$, and the multiplicative group $\mathbb{F}_{q^{12}}^\times$ has order $q^{12} - 1$. Hence there exists A of order $q^{12} - 1$ (a Singer cycle).

Conversely, every element $A \in GL_{12}(\mathbb{F}_q)$ admits a Jordan–Chevalley decomposition $A = su = us$, where s is semisimple and u is unipotent. Hence $|A| = \text{lcm}(|s|, |u|)$. The unipotent part has order a power of p , and any nontrivial unipotent Jordan block has size at least 2, so including a nontrivial unipotent part reduces the dimension available to the semisimple part and yields total order strictly less than $q^{12} - 1$.

The semisimple part decomposes into irreducible blocks of degrees d_1, \dots, d_k with $\sum d_i \leq 12$, and its order divides

$$L = \text{lcm}(q^{d_1} - 1, \dots, q^{d_k} - 1).$$

Write $q^{d_i} - 1 = (q - 1)a_i$. Then

$$L = (q - 1) \text{lcm}(a_1, \dots, a_k) \leq (q - 1) \prod_{i=1}^k a_i = \frac{\prod_{i=1}^k (q^{d_i} - 1)}{(q - 1)^{k-1}}.$$

If $k \geq 2$, then $\prod_{i=1}^k (q^{d_i} - 1) < q^{12} - 1$, hence

$$L < \frac{q^{12} - 1}{(q - 1)^{k-1}} < q^{12} - 1,$$

while if $k = 1$ and $d_1 = 12$, the order is exactly $q^{12} - 1$. Therefore the maximum element order in $GL_{12}(\mathbb{F}_q)$ is precisely $q^{12} - 1$. Applying this with $q = 13$, we may take $|B| = b = 13^{12} - 1$, which is maximal. The problem is therefore to maximize

$$b \cdot \frac{|A|}{\gcd(|A|, b)}, \quad \text{where } A \in GL_{12}(\mathbb{F}_2).$$

If A is block diagonal with irreducible blocks of degrees d_1, \dots, d_k (with $\sum d_i = 12$), then $|A| \mid \text{lcm}(2^{d_1} - 1, \dots, 2^{d_k} - 1)$. Taking blocks of degrees 5 and 7 yields $|A| = \text{lcm}(2^5 - 1, 2^7 - 1) = 31 \cdot 127 = 3937$. Since $31 \nmid b$ and $127 \nmid b$, we have $\gcd(3937, b) = 1$, so $\text{lcm}(|A|, b) = 3937b$.

We can show this as follows:

Let $A \in GL_{12}(\mathbb{F}_2)$ and write its Jordan–Chevalley decomposition $A = su$, where s is semisimple and u is unipotent. Since $b = 13^{12} - 1$ is odd, $\gcd(|u|, b) = 1$, and therefore

$$\frac{|A|}{\gcd(|A|, b)} = \frac{|s|}{\gcd(|s|, b)} \cdot |u|.$$

Thus the odd contribution comes entirely from the semisimple part s .

If s has irreducible blocks of degrees d_1, \dots, d_k with $\sum d_i \leq 12$, then $|s| \mid \text{lcm}(2^{d_1} - 1, \dots, 2^{d_k} - 1)$.

Define

$$c_d = \frac{2^d - 1}{\gcd(2^d - 1, b)}.$$

Then

$$\frac{|s|}{\gcd(|s|, b)} \leq \text{lcm}(c_{d_1}, \dots, c_{d_k}).$$

Using

$$\gcd(2^d - 1, 13^{12} - 1) = 2^{\gcd(d, 12)} - 1,$$

we obtain

$$c_d = \frac{2^d - 1}{2^{\gcd(d, 12)} - 1}.$$

For $d \leq 12$, one checks that $c_d = 1$ for $d \in \{1, 2, 3, 4, 6, 12\}$, while

$$c_5 = 31, \quad c_7 = 127, \quad c_8 = 17, \quad c_9 = 73, \quad c_{10} = 341, \quad \text{and } c_{11} = 2047.$$

Subject to $d_1 + \dots + d_k \leq 12$, the only way to combine two nontrivial contributions is $7 + 5 = 12$, giving $\text{lcm}(c_7, c_5) = \text{lcm}(127, 31) = 127 \cdot 31 = 3937$. All other feasible combinations yield a value at most $\max\{2047, 341, 73, 17, 31, 127\} < 3937$.

Therefore

$$\max_{A \in GL_{12}(\mathbb{F}_2)} \frac{|A|}{\gcd(|A|, b)} = 3937,$$

as we desired to show. This implies $N = \boxed{3937(13^{12} - 1)}$.