ICMT — Constellation Round (Division B)

Round Structure

- This is a **team** round. There are **34 problems**, divided into 5 constellations of 6 problems each, plus four other problems. Your team will have **90 minutes** total.
- Each of the 5 constellations is associated with a different subject. The subjects are Algebra, Arithmetic (Number Theory), Calculus, Combinatorics, and Probability.
- Problems are **NOT** ordered by difficulty.
- Your team may submit an answer to a problem at any time. Each team has **up to 38 submissions**.
- You may resubmit if you get a problem wrong, but only the **LATEST** submission per problem counts.
- During the round, there will be a live scoreboard displaying teams' scores and which problems they have solved (except for Problem 4). The scoreboard **will not be updated in the last 5 minutes**.

Scoring

Your team's score is the sum of the following four categories:

- **Base Points:** Each correct answer receives 15 points. Incorrect answers receive 0 points.
  - One problem (Problem 4) has a different scoring system (which will be revealed during the test), and is worth up to 75 points. It is ineligible for bonus points (explained below).
  - The maximum number of base points is $15 \cdot 33 + 75 = \mathbf{570}$.
- **Speed Bonus:** Bonus points are awarded for solving problems before other teams.
  - The N^{th} team to solve a problem will receive $\max(16 - N, 0)$ bonus points. In other words, the first team to solve a problem will receive 15 bonus points, the second team to solve a problem will receive 14 bonus points, etc. The sixteenth team and later teams will receive no bonus points.
  - We expect roughly 70 teams to be competing.
  - The maximum number of speed bonus points is $15 \cdot 33 = \mathbf{495}$.
- **Depth Bonus:** Bonus points are awarded for solving multiple problems in the same constellation.
  - Within a constellation, $3(N - 1)$ bonus points are awarded on the N^{th} correct answer, for a total bonus of $3N(N - 1)/2$ depth points per constellation.

Problems solved in constellation	0	1	2	3	4	5	6
Total Depth Bonus	0	0	3	9	18	30	45

  - The maximum number of depth bonus points is $5 \cdot 45 = \mathbf{225}$.
- **Breadth Bonus:** Bonus points are awarded for solving problems in every constellation.
  - For $N > 0$, $15N + 20$ bonus points are awarded upon solving at least N problems in each of the 5 constellations, for a total bonus of $15N(N + 1)/2 + 20N$ breadth points.

Problems solved in every constellation	0	1	2	3	4	5	6
Total Breadth Bonus	0	35	85	150	230	325	**435**

The maximum score on this test is $570 + 495 + 225 + 435 = \mathbf{1725}$ points.

Rotating Black Hole ●

1. Let A be the answer to this problem. Let $f : \mathbb{R}_{>0} \to \mathbb{R}$ be the smooth function such that $f(1) = 1$ and
$$f(x) = x^A + xf'(x).$$
Compute the positive real number r such that $f(r) = 0$.

2. Let A_3 be the answer to Problem 3. Let M be a real $A_3 \times A_3$ matrix. Counting eigenvalues with multiplicity, what is sum of all possible values of the number of real eigenvalues of M?

3. Let A_2 be the answer to Problem 2. Rafiel picks an integer from 1 to A_2, uniformly at random. If the integer is not relatively prime to A_2, Rafiel receives 0 points. Otherwise, he receives points equal to $\frac{4}{3}$ times the number he picked. What is Rafiel's expected score?

Singular Sun ⊙

4. Submit an integer from 1 to 200, inclusive. Let N be your integer, let T be the total number of teams, let B be the number of teams whose first submission of a valid answer to this problem occurred before your submission, and of those B teams, let A be the number of teams who submitted an answer that differs from N by at most 5. You will receive $\left\lfloor 75 \cdot \frac{N}{200} \cdot \frac{4}{A+4} \cdot \left(\frac{1}{5} + \frac{4}{5} \cdot \frac{B}{T}\right) \right\rfloor$ points for a valid answer, and 0 points for submitting an invalid answer or submitting more than once.

Algebra Aries ♈

5. Compute the determinant of the matrix
$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 5 & 4 & 3 \\ 1 & 4 & 25 & 16 & 9 \\ 1 & 8 & 125 & 64 & 27 \\ 1 & 16 & 625 & 256 & 0 \end{pmatrix}.$$

6. Compute the number of complex numbers z such that $|z| = 1$ and
$$z^{340} + z^{140} = -1.$$

7. Let r and s be the roots of $x^2 - x + 4$. Compute the value of
$$\sum_{n=1}^{\infty} \frac{1}{r^n} + \frac{1}{s^n}.$$

8. What is the interval of real values of c such that the equation
$$(x + y)^2 = c(x - 2026)(y + 2026)$$
has exactly one real solution (x, y)?

9. Let (x, y) be a pair of positive integers satisfying $xy^2 - y^2 - x + y = 10$. Compute the sum of all possible values of $x + y$.

10. Let $V = \mathbb{R}[X]_{\leq 2025}$ be the vector space of real polynomials of degree at most 2025. Consider the derivative map $f : V \to V$, defined by
$$f\left(\sum_{i=0}^{2025} a_i X^i\right) = \sum_{i=1}^{2025} i a_i X^{i-1},$$
where each $a_i \in \mathbb{R}$. Calculate the eigenvalue of f with the largest magnitude.

Arithmetic Aquarius ♒

11. The Fibonacci sequence $\{F_n\}$ is defined recursively by $F_1 = 1$, $F_2 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 1$. Call a positive integer k *repetitive* if and only if there exists a positive integer n such that k divides both F_n and F_{n+6}. Compute the sum of all repetitive positive integers.

12. Over all ordered pairs of prime numbers (p, q) such that

$$pq \mid p^{q^2} + q^{2p^2} + 1,$$

compute the sum of all possible values of $p + q$.

13. How many ordered integer triples, (a, b, c), with $1 \leq a, b, c \leq 40$, are there such that a, b, and c form the side lengths of a triangle and $a^2 + ab = c^2$?

14. Compute the largest integer n for which $\varphi(n) \leq 10$.

15. Compute the sum of all rational values of x such that $2x^2 - 19x + 42$ is prime.

16. A polynomial $P(x)$ is called *nice* if $P(1)$ and $P(-1)$ are both divisible by 11, and $P(x)$ is called *small* if all of its coefficients are integers with absolute value at most 5. Calculate the number of nice small polynomials of degree at most 4 (counting $P(x) = 0$).

Calculus Capricorn ♑

17. There is a unique line that is tangent to the curve $y = x^4 - 8x^3 + 22x^2 - 20x + 26$ at two distinct points and that does not intersect the curve at any other points. Given that the tangent line is of the form $y = ax + b$, compute (a, b).

18. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $f'(x) = f(x) - x$ for all real x and $f(x) > 0$ for all $x > -2$. What is the minimum possible value of $f(0)$?

19. Compute the limit:

$$\lim_{x \to 0} \frac{\ln\bigl(1 + \ln\bigl(1 + \ln\bigl(1 + x^{2026}\bigr)\bigr)\bigr)}{x^{2025} \sin(x + \sin(x + \sin(x)))}.$$

20. Compute the value of

$$\int_0^\infty \frac{dx}{x^4 + 5x^2 + 4}.$$

21. Compute the sum

$$\sum_{n=0}^{\infty} \frac{27(-1)^n}{9n^2 + 15n + 4}.$$

22. Suppose x, y, z are real numbers such that $x + 2y + z = 3$ and $2x + y - z = 6$. Compute the minimum possible value of $4x^2 + y^2 + 3z^2$.

Combinatorics Cancer ♋

23. For a permutation σ of 100 elements, let $m(\sigma)$ be the minimum number of swaps of adjacent elements needed to return σ to the identity permutation. What is the maximum of $m(\sigma)$ over all permutations of 100 elements σ?

24. How many subsets of $\{1, 2, \ldots, 10\}$ are there such that for any three consecutive integers, at most one of them lies in the subset?

25. Suppose mn points are arranged in an $m \times n$ lattice. Suppose we place unit masses on x of the lattice points (placing at most one mass at each point). Let $f(m, n)$ be the number of positive integers x for which it is possible to place the masses so that their centroid, defined as the average of the position vectors of the chosen points, is equal to the centroid of all mn lattice points. Compute
$$\sum_{m=1}^{10} \sum_{n=1}^{10} f(m, n).$$

26. Rohit is traveling on the (x, y)-plane, starting at $(0, 0)$, and can only move in the following two ways:
$$(x, y) \mapsto (x + 3, y + 2), \qquad (x, y) \mapsto (x - 3, y - 1).$$
How many possible sequences of moves allow Rohit to reach the point $(6, 7)$?

27. Ten people are attending a paintball tournament and will be assigned to two opposing teams of 5: Red Team and Blue Team. (The teams are labeled, so swapping Red and Blue counts as a different team selection.) However, some people are enemies with each other and do not want to be on the same team. For a given set of pairs of enemies E, let $f(E)$ be the number of "valid team selections" with no enemy pairs on the same team. What is the sum of $f(E)$ over all possible sets of enemy pairs?

28. There are six people living on the same floor of a dorm, where the rooms are numbered $1, 2, \ldots, 6$ (each student lives in a different room). Each pair of residents is either friends or enemies. For every triple of integers (i, j, k) with $1 \leq i < j < k \leq 6$, the following are true:
   - If the resident in room j is friends with the residents in both room i and room k, then each pair among the three are friends.
   - If the resident in room j is enemies with the residents in both room i and room k, then each pair among the three are enemies.

   How many possible friendship configurations are there among these six people?

Probability Pisces ♓

29. A cube of side length 2026 is painted on the exterior and is then cut into 2026^3 unit cubes. A unit cube is then chosen uniformly at random and is rolled twice independently. Given that the top face on the first roll is painted, what is the probability that the top face on the second roll is also painted?

30. Kaity is playing a number game. She writes the number 324000 on a whiteboard, and at each step she chooses one of the positive divisors of the most recently written number uniformly at random and writes it on the whiteboard. She repeats this process until she writes the number 1. What is the expected value of the natural logarithm of the product of all the numbers written on the whiteboard, including the initial number 324000?

31. A 4-sided die, with faces numbered 1, 2, 3, and 4, is rolled repeatedly and independently until a 2 is rolled. What is the probability that the sum of the rolls (including the last roll) is divisible by 3?

32. Cat and David are playing a game. They begin with the number 10 written on a whiteboard. The two then take turns, starting with Cat. On a player's turn, the player chooses a number x uniformly at random from the interval $[0, 1]$, independently of all previous choices, and multiplies the number on the whiteboard by x. They then replace the number on the whiteboard with this product. The game continues until the number on the whiteboard is at most 1. What is the probability that Cat is the last person to take a turn?

33. Let A, B, and C be points chosen independently and uniformly at random on the unit circle centered at O. Let G be the centroid (center of mass) of triangle $\triangle ABC$. What is the expected value of GO^2?

34. Choose 10 numbers $X_1, ..., X_{10}$ independently and uniformly at random from the interval $[0, 1]$. What is the expected value of $\min(X_1, ..., X_{10})$?