ICMT — Constellation Round Solutions (Division B)

Rotating Black Hole ●

1. Let A be the answer to this problem. Let $f : \mathbb{R}_{>0} \to \mathbb{R}$ be the smooth function such that $f(1) = 1$ and
$$f(x) = x^A + xf'(x).$$
Compute the positive real number r such that $f(r) = 0$.

Answer: 2

Solution: Taking the derivative of the equation gives that
$$f'(x) = Ax^{A-1} + f'(x) + xf''(x),$$
so
$$f''(x) = -Ax^{A-2}.$$
We have three cases:

Case 1: $A = 0$. In this case $f''(x) = 0$, so $f'(x) = c$, and $f(x) = cx + d$ for some constants c, d. We have that
$$cx + d = f(x) = x^A + xf'(x) = 1 + xc,$$
so $d = 1$, and $f(1) = 1$ gives that $c = 0$. Hence $f(x) = 1$, and r does not exist in this case.

Case 2: $A = 1$. Then $f(r) = f(A) = f(1) = 0$, which is not possible because we are given that $f(1) = 1$.

Case 3: $A \neq 0, 1$. Then $f'(x) = -\frac{A}{A-1}x^{A-1} + c$, and $f(x) = -\frac{1}{A-1}x^A + cx + d$ for some constants c, d. We have that
$$-\frac{1}{A-1}x^A + cx + d = f(x) = x^A + xf'(x) = x^A + x\left(-\frac{A}{A-1}x^{A-1} + c\right),$$
and these are equal when $d = 0$. $f(1) = 1$ gives that $f(x) = -\frac{1}{A-1}x^A + \frac{A}{A-1}x$, and so $r = A^{\frac{1}{A-1}}$ for $A > 0$. In particular, we note that r does not exist for $A < 0$.

Since $A \leq 0$ and 1 do not give valid solutions, we know that $A > 0$, with $A \neq 1$, and
$$A^{\frac{1}{A-1}} = A.$$
Solving gives that $A = \boxed{2}$.

2. Let A_3 be the answer to Problem 3. Let M be a real $A_3 \times A_3$ matrix. Counting eigenvalues with multiplicity, what is sum of all possible values of the number of real eigenvalues of M?

Answer: 72

Solution: The number of real eigenvalues is equal to the number of possible real roots to the characteristic polynomial of M, which is a degree A_3 polynomial. Since the number of complex, non-real roots must be even (they come in conjugate pairs), the number of possible real eigenvalues is thus $A_3 - 2k$, for $0 \leq k \leq \lfloor A_3/2 \rfloor$. Hence, if A_3 is even, the answer to this problem is $(A_3/2)(A_3/2+1)$, and if A_3 is odd, the answer to this problem is $((A_3+1)/2)^2$.

Moreover, we must have that $A_3 = 2\varphi(A_2)/3$ by problem 3. In other words, if $A_3 = 2k$ is even, then we need that $\varphi(k(k+1)) = 3k$, and if $A_3 = 2k-1$ is odd, then $\varphi(k^2) = (6k-3)/2$. One can verify that this has no solutions in this case.

Now for $A_3 = 2k$ even, we wish to solve the equation $\varphi(k(k+1)) = \varphi(k)\varphi(k+1) = 3k$. In particular, note that
$$v_2(\varphi(k)\varphi(k+1)) \geq v_2(\varphi(k)) + 1 \geq v_2(k) + \text{ number of distinct odd primes dividing } k.$$
Thus, to get the equality $\varphi(k)\varphi(k+1) = 3k$, there cannot be any odd prime factors dividing k. Thus, k must be a power of 2. One can verify that the only possible solution is $k = 8$, so $A_3 = 16$ and $A_2 = \boxed{72}$.

3. Let A_2 be the answer to Problem 2. Rafiel picks an integer from 1 to A_2, uniformly at random. If the integer is not relatively prime to A_2, Rafiel receives 0 points. Otherwise, he receives points equal to $\frac{4}{3}$ times the number he picked. What is Rafiel's expected score?

   Answer: 16

   Solution: The expected score can be computed to be

$$\sum_{\substack{\gcd(i,n)=1 \\ 1 \le i \le n}} \frac{4i}{3n} = \frac{2\varphi(n)}{3}.$$

   Combining with Problem 2, we have that $A_2 = 72$, so the answer to this problem is $\boxed{16}$.

Singular Sun ⊙

4. Submit an integer from 1 to 200, inclusive. Let N be your integer, let T be the total number of teams, let B be the number of teams whose first submission of a valid answer to this problem occurred before your submission, and of those B teams, let A be the number of teams who submitted an answer that differs from N by at most 5. You will receive $\left\lfloor 75 \cdot \frac{N}{200} \cdot \frac{4}{A+4} \cdot \left(\frac{1}{5} + \frac{4}{5} \cdot \frac{B}{T}\right) \right\rfloor$ points for a valid answer, and 0 points for submitting an invalid answer or submitting more than once.

   Solution: N/A

Algebra Aries ♈

5. Compute the determinant of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 5 & 4 & 3 \\ 1 & 4 & 25 & 16 & 9 \\ 1 & 8 & 125 & 64 & 27 \\ 1 & 16 & 625 & 256 & 0 \end{pmatrix}.$$

   Answer: 5544

   Solution: Notice that the matrix above is almost Vandermonde. Thus, one can use the standard Vandermonde determinant, then apply an adjustment to subtract terms that are associated with the bottom right element. Calling the top-left 4×4 matrix as V_4, the whole matrix as A, and

$$V_5 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 5 & 4 & 3 \\ 1 & 4 & 25 & 16 & 9 \\ 1 & 8 & 125 & 64 & 27 \\ 1 & 16 & 625 & 256 & 81 \end{pmatrix}$$

   Then we note that $\det A = \det V_5 - 81 \det V_4$. By the standard formula,

$$\det V_5 = (2-1)(3-1)(4-1)(5-1)(3-2)(4-2)(5-2)(4-3)(5-3)(5-4) = 288$$

   and

$$\det V_4 = -(2-1)(4-1)(5-1)(4-2)(5-2)(5-4) = -72$$

   Thus, the determinant is $\det A = 288 - 81 \cdot (-72) = \boxed{5544}$.

6. Compute the number of complex numbers z such that $|z| = 1$ and

$$z^{340} + z^{140} = -1.$$

Answer: 40

Solution: Let $w = z^{20}$, so that we want to find complex numbers satisfying $w^{17} + w^7 + 1 = 0$. We need $w = e^{i\theta}$ since the magnitude of w is $|z|^{20} = 1$. Then, in order for $e^{i\theta} + e^{i\phi} = -1$, we need $\theta, \phi = \frac{2\pi}{3}, \frac{4\pi}{3}$ in some order (i.e. $e^{i\theta}, e^{i\phi} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$.) This gives $7\theta, 17\theta = \frac{2\pi}{3} + 2\pi k, \frac{4\pi}{3} + 2\pi l$ in some order. We get $3\theta = \pm 2\pi(l - 2k)$, which either fixes $\theta = \frac{2\pi}{3} + 2\pi x$ or $\theta = \frac{4\pi}{3} + 2\pi x$. Thus, our two solutions for w are $w = e^{2\pi i/3}, e^{4\pi i/3}$. Then, $z^{20} = w$ has 20 unique solutions for each of those cases, giving a total of $2 \cdot 20 = \boxed{40}$ possible values of z.

7. Let r and s be the roots of $x^2 - x + 4$. Compute the value of

$$\sum_{n=1}^{\infty} \frac{1}{r^n} + \frac{1}{s^n}.$$

Answer: $-\frac{1}{4}$

Solution: The geometric series formula gives

$$\sum_{n=1}^{\infty} \frac{1}{r^n} + \frac{1}{s^n} = \frac{1}{r-1} + \frac{1}{s-1} = \frac{r+s-2}{(1-r)(1-s)}.$$

By Vieta's, $r + s = 1$, and $(1-r)(1-s) = 1^2 - 1 + 4 = 4$, giving an answer of $\boxed{-\frac{1}{4}}$.

8. What is the interval of real values of c such that the equation

$$(x+y)^2 = c(x - 2026)(y + 2026)$$

has exactly one real solution (x, y)?

Answer: $(0, 4)$

Solution: Making the substitution $u = x - 2026$ and $v = y + 2026$, this is equivalent to $(u+v)^2 = cuv$, or $u^2 + (2-c)uv + v^2 = 0$.

We will always have the real solution $(u, v) = (0, 0)$. When $v \neq 0$, we may also gain the real solutions $\frac{u}{v} = r$, if r is a real root of the quadratic

$$r^2 + (2-c)r + 1 = 0.$$

We only want a single real solution, so we want there to be no such real r. This occurs precisely when $(2-c)^2 - 4 < 0$, or when $c \in \boxed{(0, 4)}$.

9. Let (x, y) be a pair of positive integers satisfying $xy^2 - y^2 - x + y = 10$. Compute the sum of all possible values of $x + y$.

Answer: 23

Solution: We can write that

$$x = \frac{10 - y + y^2}{y^2 - 1}.$$

Since $x \geq 1$, we must have that $y \leq 11$. Hence $1 \leq y \leq 11$, so we can check all cases. There are 3 solutions: $(1, 11)$, $(2, 3)$, and $(4, 2)$, which sum to $\boxed{23}$.

10. Let $V = \mathbb{R}[X]_{\leq 2025}$ be the vector space of real polynomials of degree at most 2025. Consider the derivative map $f : V \to V$, defined by

$$f\left(\sum_{i=0}^{2025} a_i X^i\right) = \sum_{i=1}^{2025} i a_i X^{i-1},$$

where each $a_i \in \mathbb{R}$. Calculate the eigenvalue of f with the largest magnitude.

Answer: 0

Solution: All eigenvalues of f are $\boxed{0}$, since the derivative of any polynomial can't be a scalar multiple of it. Another way to see this is as follows: note that f is nilpotent, i.e. applying f sufficiently many times to any element of R makes it 0. Consequently all eigenvalues of f must be 0.

Arithmetic Aquarius ≈

11. The Fibonacci sequence $\{F_n\}$ is defined recursively by $F_1 = 1$, $F_2 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 1$. Call a positive integer k *repetitive* if and only if there exists a positive integer n such that k divides both F_n and F_{n+6}. Compute the sum of all repetitive positive integers.

Answer: 15

Solution: Note that

$$F_{n+6} = F_{n+5} + F_{n+4} = 2F_{n+4} + F_{n+3} = 3F_{n+3} + 2F_{n+2} = 5F_{n+2} + 3F_{n+1} = 8F_{n+1} + 5F_n.$$

Now, if k divides both F_{n+6} and F_n, then we see that it divides $8F_{n+1}$. However, F_{n+1} and F_n are always coprime, so we conclude that k must divide 8. All such k work, as can be seen by taking any n divisible by 6. The answer is thus $1 + 2 + 4 + 8 = \boxed{15}$.

12. Over all ordered pairs of prime numbers (p, q) such that

$$pq \mid p^{q^2} + q^{2p^2} + 1,$$

compute the sum of all possible values of $p + q$.

Answer: 20

Solution:

Taking mod q, we have that $p^{q^2} \equiv -1 \pmod{q}$. If q is 2, we conclude that p must be odd. If q is an odd prime, the order of p mod q must divide $2q^2$ and must not divide q^2. Moveover, by Fermat's Little Theorem, the order must be at most $q - 1$. Hence the order of p mod q is 2. Thus, either $q = 2$ and p is odd, or q is odd and $q \mid p + 1$.

Taking mod p, we have that $q^{2p^2} \equiv -1 \pmod{p}$. If p is 2, we conclude that q must be odd. If p is an odd prime, the order of q mod p must divide $4p^2$ and must not divide $2p^2$, and must be at most $p - 1$. Hence the order of q mod p is 4. Thus, either $p = 2$ and q is odd, or p is odd and $p \mid q^2 + 1$.

We now evaluate all possible cases:

Case 1: $p = 2$. Then q is an odd prime with $q \mid p + 1$, hence $q = 3$.

Case 2: $q = 2$. Then p is an odd prime with $p \mid q^2 + 1$, hence $p = 5$.

Case 3: p and q are odd primes. Then $q \mid p + 1$ and $p \mid q^2 + 1$. In particular, we can write $p + 1 = kq$, for some positive integer $k < p + 1$. Since $q^2 \equiv -1 \pmod{p}$, we get that $k \equiv -q \pmod{p}$, and hence $k = p - q$. Thus, we are looking for solutions to $p + 1 = q(p - q)$. Solving gives that $p = \frac{q^2+1}{q-1}$; since $\gcd(q^2 + 1, q - 1) = 2$, this is only possible if $q = 3$ and $p = 5$.

This exhausts all possibilities, and thus the sum of all possible $p + q$ is $5 + 7 + 8 = \boxed{20}$.

13. How many ordered integer triples, (a, b, c), with $1 \le a, b, c \le 40$, are there such that a, b, and c form the side lengths of a triangle and $a^2 + ab = c^2$?

Answer: 17

Solution: The difficulty in this problem lies in parameterizing and efficiently iterating through all possible solutions. We outline a strategy below.

For now, we assume that a, b are relatively prime. Then we can parameterize all solutions in the form $a = x^2$, $b = y^2 - x^2$, and $c^2 = (xy)^2$, with $\gcd(x, y) = 1$. Since a, b, and c form a triangle, the triangle inequality also gives relation $x < y < 2x$.

We now can iterate through all possible values of xy up to 40, with $\gcd(x, y) = 1$ and $x < y < 2x$. To get the non relatively prime solutions, we can scale x and y by a constant multiple. Iterating all possible solutions gives $\boxed{17}$ solutions: (4, 5, 6), (8, 10, 12), (9, 7, 12), (9, 16, 15), (12, 15, 18), (16, 9, 20), (16, 20, 24), (18, 14, 24), (16, 33, 28), (18, 32, 30), (20, 25, 30), (25, 11, 30), (25, 24, 35), (24, 30, 36), (27, 21, 36), (25, 39, 40), (32, 18, 40).

14. Compute the largest integer n for which $\varphi(n) \le 10$.

Answer: 30

Solution: To begin, note that n cannot admit a prime factor larger than 11: the surjection $(\mathbb{Z}/n\mathbb{Z})^\times \to (\mathbb{Z}/p\mathbb{Z})^\times$ reveals that $(\mathbb{Z}/n\mathbb{Z})^\times$ has at least $p - 1 > 10$ elements! Thus,

$$\frac{\varphi(n)}{n} \ge \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \frac{10}{11} = \frac{16}{77} > \frac{1}{5},$$

so $n < 50$, and similarly $n < 25$ if n is odd. Some quick casework reveals that $n = \boxed{30}$ is the largest possible value.

15. Compute the sum of all rational values of x such that $2x^2 - 19x + 42$ is prime.

Answer: 19

Solution: We can factor $2x^2 - 19x + 42 = (x - 6)(2x - 7)$. Let $x = a/b$, with $\gcd(a, b) = 1$ and $b > 0$. We have then that

$$(a - 6b)(2a - 7b) = pb^2$$

for some prime b. Since $\gcd(a, b) = 1$ and b divides the LHS, we need that $b \mid 2a - 7b$, and hence $b \mid 2a$. Thus, $b = 1$ or $b = 2$.

If x is integer ($b = 1$), then we need that $x - 6 = \pm 1$ or $2x - 7 = \pm 1$. Checking cases gives that $x = 3$ and $x = 7$ are valid.

If x is of the form $\frac{a}{2}$, for a odd, then we need that $(a - 12)(a - 7) = 2p$ for some prime p. We can check that $a = 5$ and $a = 13$ are the only valid solutions. Thus, the sum of all solutions is $3 + 7 + \frac{5}{2} + \frac{13}{2} = \boxed{19}$.

16. A polynomial $P(x)$ is called *nice* if $P(1)$ and $P(-1)$ are both divisible by 11, and $P(x)$ is called *small* if all of its coefficients are integers with absolute value at most 5. Calculate the number of nice small polynomials of degree at most 4 (counting $P(x) = 0$).

Answer: 1331

Solution: Notice that the set of nice polynomials can be identified with $\mathbb{F}_{11}^5$. Furthermore, the set of polynomials which satisfy $P(1) = P(-1) = 0$ form a co-dimension 2 subspace of this vector space. Thus the number of nice polynomials is $11^{5-2} = \boxed{1331}$.

Calculus Capricorn ♑

17. There is a unique line that is tangent to the curve $y = x^4 - 8x^3 + 22x^2 - 20x + 26$ at two distinct points and that does not intersect the curve at any other points. Given that the tangent line is of the form $y = ax + b$, compute (a, b).

Answer: $(4, 17)$

Solution: Let $f(x)$ be the polynomial. Then for some real numbers a and b, we want $f(x) - ax - b$ to have two different double roots s and t. By Vieta's formulas, we want $2s+2t = 8$ and $s^2+t^2+4st = 22$. Solving for s and t gives the solutions of 1 and 3. Then

$$ax + b = (x^4 - 8x^3 + 22x^2 - 20x + 26) - ((x-1)(x-3))^2 = 4x + 17,$$

giving the answer $\boxed{(4, 17)}$.

18. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $f'(x) = f(x) - x$ for all real x and $f(x) > 0$ for all $x > -2$. What is the minimum possible value of $f(0)$?

Answer: $e^2 + 1$

Solution: To solve $f(x) - f'(x) = x$, we find some solution to $f(x) - f'(x) = x$ as a polynomial, and then we can add Ce^x with some constant C as a solution to $f(x) - f'(x) = 0$ that won't change the relation. Plugging in $f(x) = ax + b$, we get $f(x) - f'(x) = ax + (b - a) = x$, meaning $a = b = 1$, so $f(x) = x + 1 + Ce^x$.

Then $f(0) = C + 1$, so minimizing $f(0)$ corresponds to minimizing C. First, we note that C must be positive, as we need that $f(x)$ is positive as $x \to \infty$. Then minimizing C corresponds to minimizing the value of $f(-2)$, since e^x is always positive. Since $f(x) > 0$ for all $x > -2$, the best we can do is setting $f(-2) = 0$. Therefore $f(-2) = 0 = -1 + Ce^{-2}$ giving $C = e^2$, and $f(0) = \boxed{e^2 + 1}$.

19. Compute the limit:

$$\lim_{x \to 0} \frac{\ln\big(1 + \ln\big(1 + \ln\big(1 + x^{2026}\big)\big)\big)}{x^{2025} \sin(x + \sin(x + \sin(x)))}.$$

Answer: $\frac{1}{3}$

Solution: Since $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$, observe that

$$\lim_{x \to 0} \frac{\sin(x + \sin(x + \sin(x)))}{x} = \lim_{x \to 0} \frac{x + \sin(x + \sin(x))}{x} = 1 + \lim_{x \to 0} \frac{x + \sin(x)}{x} = 3.$$

Since $\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$, observe that

$$\lim_{x \to 0} \frac{\ln\big(1 + \ln\big(1 + \ln\big(1 + x^{2026}\big)\big)\big)}{x^{2026}} = \lim_{x \to 0} \frac{\ln\big(1 + \ln\big(1 + x^{2026}\big)\big)}{x^{2026}} = \lim_{x \to 0} \frac{\ln\big(1 + x^{2026}\big)}{x^{2026}} = 1.$$

Thus,

$$\lim_{x \to 0} \frac{\ln\big(1 + \ln\big(1 + \ln\big(1 + x^{2026}\big)\big)\big)}{x^{2025} \sin(x + \sin(x + \sin(x)))} = \lim_{x \to 0} \frac{\ln\big(1 + \ln\big(1 + \ln\big(1 + x^{2026}\big)\big)\big)}{x^{2026}} \cdot \frac{x}{\sin(x + \sin(x + \sin(x)))} = \boxed{\frac{1}{3}}.$$

20. Compute the value of

$$\int_0^\infty \frac{dx}{x^4 + 5x^2 + 4}.$$

Answer: $\frac{\pi}{12}$

Solution: We have the partial fraction decomposition

$$\frac{1}{(x^2+1)(x^2+4)} = \frac{\frac{1}{3}}{x^2+1} - \frac{\frac{1}{3}}{x^2+4}.$$

Hence

$$\begin{aligned} \int_0^\infty \frac{dx}{x^4 + 5x^2 + 4} &= \frac{1}{3} \int_0^\infty \frac{dx}{x^2+1} - \frac{1}{3} \int_0^\infty \frac{dx}{x^2+4} \\ &= \frac{1}{3} \arctan(x) \Big|_0^\infty - \frac{1}{6} \arctan\left(\frac{x}{2}\right) \Big|_0^\infty \\ &= \boxed{\frac{\pi}{12}}. \end{aligned}$$

21. Compute the sum

$$\sum_{n=0}^{\infty} \frac{27(-1)^n}{9n^2+15n+4}.$$

Answer: $2\sqrt{3}\pi - 9 + 6\ln(2)$

Solution: We can rewrite the sum in the form

$$\begin{aligned} 9\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{3n+1} - \frac{(-1)^n}{3n+4}\right) =& 9\sum_{n=0}^{\infty}(-1)^n\left(\int_0^1 x^{3n}\,dx - \int_0^1 x^{3n+3}dx\right) \\ =& 9\int_0^1 \frac{1-x^3}{1+x^3}\,dx. \end{aligned}$$

The integral can now be evaluated by partial fractions; in particular,

$$\int \frac{1-x^3}{1+x^3}\,dx = -x - \frac{1}{3}\log\bigl(x^2-x+1\bigr) + \frac{2}{3}\log(x+1) + \frac{2\sqrt{3}}{3}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C.$$

Evaluating at the bounds gives the answer $\boxed{2\sqrt{3}\pi - 9 + 6\ln(2)}$.

22. Suppose x, y, z are real numbers such that $x + 2y + z = 3$ and $2x + y - z = 6$. Compute the minimum possible value of $4x^2 + y^2 + 3z^2$.

Answer: 18

Solution: We use the method of Lagrange multipliers. We are minimizing $4x^2 + y^2 + 3z^2$ subject to the constraints $x + 2y + z - 3 = 0$ and $2x + y - z - 6 = 0$. Taking gradients and setting up the system gives

$$\begin{aligned} 8x &= \lambda + 2\mu \\ 2y &= 2\lambda + \mu \\ 6z &= \lambda - \mu \\ 3 &= x + 2y + z \\ 6 &= 2x + y - z \end{aligned}$$

Adding the first and third equations gives

$$8x + 6z = 2\lambda + \mu$$

so we can solve the system of three equations

$$\begin{aligned} 2y &= 8x + 6z \\ 3 &= x + 2y + z \\ 6 &= 2x + y - z \end{aligned}$$

Dividing the first equation by 2 and substituting y in the other two equations gives

$$\begin{aligned} 3 &= 9x + 7z \\ 6 &= 6x + 2z \end{aligned}$$

which solves to $z = \frac{-3}{2}, x = \frac{3}{2}$. Finally we get $y = 4x + 3z = \frac{3}{2}$, and our final answer is $4x^2 + y^2 + 3z^2 = 8 \cdot \frac{3^2}{2^2} = \boxed{18}$.

Combinatorics Cancer ♋

23. For a permutation σ of 100 elements, let $m(\sigma)$ be the minimum number of swaps of adjacent elements needed to return σ to the identity permutation. What is the maximum of $m(\sigma)$ over all permutations of 100 elements σ?

Answer: 4950

Solution: We claim that for a permutation of n elements, the maximum of $m(\sigma)$ is $n(n-1)/2$; this gives an answer of $\boxed{4950}$.

To show that $n(n-1)/2$ is sufficient: Insertion sort the permutation. This causes at most $n(n-1)/2$ adjacent element swaps before the list is sorted.

To show that $n(n-1)/2$ is necessary: Define an inversion as two indices i, j such that $\sigma_i > \sigma_j$. Note that $n(n-1)/2$ inversions exist in the permutation $(n, n-1, ..., 2, 1)$. Each swap of adjacent elements reduces the number of inversions by at most one. Therefore, sorting the permutation $(n, n-1, ..., 2, 1)$ requires at least $n(n-1)/2$ swaps.

24. How many subsets of $\{1, 2, \dots, 10\}$ are there such that for any three consecutive integers, at most one of them lies in the subset?

Answer: 60

Solution: We solve the problem recursively; let F_n be the number of such subsets of $\{1, 2, \dots, n\}$. We note that $F_1 = 2$, $F_2 = 3$, and $F_3 = 4$. Now we compute F_n for $n > 3$. If n lies in our subset, then the rest of a subset is one of the F_{n-3} possible subsets of $\{1, 2, \dots, n-3\}$. If n does not lie in our subset, then the rest of the subset is one of the F_{n-1} possible subsets of $\{1, 2, \dots, n-1\}$. Hence, we get the recurrence relation

$$F_n = F_{n-1} + F_{n-3}$$

and solving gives $F_{10} = \boxed{60}$.

25. Suppose mn points are arranged in an $m \times n$ lattice. Suppose we place unit masses on x of the lattice points (placing at most one mass at each point). Let $f(m, n)$ be the number of positive integers x for which it is possible to place the masses so that their centroid, defined as the average of the position vectors of the chosen points, is equal to the centroid of all mn lattice points. Compute

$$\sum_{m=1}^{10} \sum_{n=1}^{10} f(m, n).$$

Answer: 1825

Solution: We can always balance an even number of points by placing them symmetrically across the center of the lattice. If both m and n are odd, we can also balance an odd number of points by placing one point at the center of the lattice and balancing the remaining even number as described before.

However, if m or n is even, we can draw a line of symmetry between the slots that does not intersect any slot. We note that there must be a different parity of slots on either side of the line. This implies that the sum of the distances to the line of symmetry on one side of the line is a half-integer where the other side of the line is an integer. This makes it impossible for the sum of all distances to the line of symmetry to be 0, which means the points cannot be balanced.

Therefore, if m, n are both odd we have mn many solutions, but otherwise we have $\frac{mn}{2}$ solutions. We then sum

$$\sum_{m=1}^{10} \sum_{n=1}^{10} \left(\frac{mn}{2}\right) + \sum_{a=1}^{5} \sum_{b=1}^{5} \frac{(2a-1)(2b-1)}{2}$$

This comes out to $\frac{1}{2} \cdot 55^2 + \frac{1}{2} \cdot 25^2 = \boxed{1825}$.

26. Rohit is traveling on the (x, y)-plane, starting at $(0, 0)$, and can only move in the following two ways:

$$(x, y) \mapsto (x + 3, y + 2), \qquad (x, y) \mapsto (x - 3, y - 1).$$

How many possible sequences of moves allow Rohit to reach the point $(6, 7)$?

Answer: 56

Solution: If Rohit makes a of the first move and b of the second move, to get to $(6, 7)$ we must have that $3a - 3b = 6$ and $2a - b = 7$. This has the solution $(a, b) = (5, 3)$, so any path Rohit makes to $(6, 7)$ must consist of 5 of the first move and 3 of the second. Any ordering of these 8 moves is valid, giving $\binom{8}{3} = \boxed{56}$ possible paths.

27. Ten people are attending a paintball tournament and will be assigned to two opposing teams of 5: Red Team and Blue Team. (The teams are labeled, so swapping Red and Blue counts as a different team selection.) However, some people are enemies with each other and do not want to be on the same team. For a given set of pairs of enemies E, let $f(E)$ be the number of "valid team selections" with no enemy pairs on the same team. What is the sum of $f(E)$ over all possible sets of enemy pairs?

Answer: $63 \cdot 2^{27}$

Solution: We multiply the probability that two teams with random enemy pairs are valid by the number of possible choices of enemy pairs and choices of teams. Each team has $\binom{5}{2} = 10$ internal pairs that must not be enemies, so there are 20 restrictions which are equally likely to be satisfied as unsatisfied. So the probability of two teams with random enemy pairs being a valid team selection is $\frac{1}{2^{20}}$, while there are 2^{45} possible choices of enemy pairs (45 binary choices). There are $\binom{10}{5}$ choices of teams, by simply choosing the red team and forcing the blue team. Therefore, the answer is $\binom{10}{5} \cdot 2^{25} = \boxed{63 \cdot 2^{27}}$.

28. There are six people living on the same floor of a dorm, where the rooms are numbered $1, 2, \ldots, 6$ (each student lives in a different room). Each pair of residents is either friends or enemies. For every triple of integers (i, j, k) with $1 \le i < j < k \le 6$, the following are true:

- If the resident in room j is friends with the residents in both room i and room k, then each pair among the three are friends.
- If the resident in room j is enemies with the residents in both room i and room k, then each pair among the three are enemies.

How many possible friendship configurations are there among these six people?

Answer: 720

Solution: One can show that there is a bijection between friendship configurations and permutations of $\{1, \ldots, 6\}$, where for every pair (i, j) with $i < j$, i and j are friends if and only if i comes before j in the permutation. This gives the answer $6! = \boxed{720}$.

Probability Pisces ♓

29. A cube of side length 2026 is painted on the exterior and is then cut into 2026^3 unit cubes. A unit cube is then chosen uniformly at random and is rolled twice independently. Given that the top face on the first roll is painted, what is the probability that the top face on the second roll is also painted?

Answer: $\frac{1015}{6078}$

Solution: Instead of rolling an individual unit cube, one can imagine cutting the 2026 length cube into 2026^3 unit cubes, picking one of the unit cubes, and gluing all the cubes back together in the same orientation, leaving everything together as one giant cube. Then, one can roll the giant cube, and the side facing up for the unit cube is the face that is rolled. In this case, the condition of the first roll of the unit cube being blue is equivalent the cube being selected being one of the 2026^2 cubes that have a blue face facing up after rolling the giant cube, and all 2026^2 cubes are equally likely to be chosen.

Of these remaining cubes, we then can split them apart and roll our selected unit cube once more. There are $6 \cdot 2026^2$ total possible faces that can roll on top, and $2026^2 + 4 \cdot 2026$ possible blue faces. Each face is equally likely to be rolled, giving a final probability of

$$\frac{2026^2 + 4 \cdot 2026}{6 \cdot 2026^2} = \boxed{\frac{1015}{6078}}.$$

30. Kaity is playing a number game. She writes the number 324000 on a whiteboard, and at each step she chooses one of the positive divisors of the most recently written number uniformly at random and writes it on the whiteboard. She repeats this process until she writes the number 1. What is the expected value of the natural logarithm of the product of all the numbers written on the whiteboard, including the initial number 324000?

    Answer: $10 \ln 2 + 8 \ln 3 + 6 \ln 5$

    Solution: We begin by solving the problem in the case that the starting number is p^n. Let P_m denote the desired expected value given that the starting number is m. Then for any positive integer k, we get the recurrence relation

$$P_{p^k} = k \log p + \frac{1}{k+1} \sum_{i=0}^{k} P_{p^i}.$$

    Solving the recurrence gives $P_{p^n} = 2n \log p$.

    Now, note that $324000 = 2^5 \cdot 3^4 \cdot 5^3$. Since the logarithm of a product is the sum of the logarithms of the individual multiplicands, we have that

$$P_{324000} = P_{2^5} + P_{3^4} + P_{5^3}.$$

    By our computation above, we conclude that the desired answer is $\boxed{10 \ln 2 + 8 \ln 3 + 6 \ln 5}$.

31. A 4-sided die, with faces numbered 1, 2, 3, and 4, is rolled repeatedly and independently until a 2 is rolled. What is the probability that the sum of the rolls (including the last roll) is divisible by 3?

    Answer: $\frac{6}{19}$

    Solution: First, we notice that rolling a 3 doesn't affect the sum modulo 3. Therefore, we can consider only the roll results of $1, 2$, and 4. Noting that the other two results are both 1 modulo 3, we need a number of results being 1 or 4 which is 1 modulo 3. This gives a final answer of

$$\left(\frac{2}{3}\right)^1 \cdot \frac{1}{3} + \left(\frac{2}{3}\right)^4 \cdot \frac{1}{3} + \ldots = \frac{3}{4} \cdot \left(\left(\frac{8}{27}\right)^1 + \left(\frac{8}{27}\right)^2 + \ldots\right) = \boxed{\frac{6}{19}}.$$

32. Cat and David are playing a game. They begin with the number 10 written on a whiteboard. The two then take turns, starting with Cat. On a player's turn, the player chooses a number x uniformly at random from the interval $[0, 1]$, independently of all previous choices, and multiplies the number on the whiteboard by x. They then replace the number on the whiteboard with this product. The game continues until the number on the whiteboard is at most 1. What is the probability that Cat is the last person to take a turn?

    Answer: $\frac{101}{200}$

    Solution: Let $f(x)$ be the probability that the first player wins, starting from a given x for $x > 1$ (when $x \leq 1$, $f(x) = 1$). Then, with probability $\frac{1}{x}$ the player will immediately win. Else, we want the other player to lose which implies that

$$f(x) = \frac{1}{x} + \int_{\frac{1}{x}}^{1} (1 - f(ux)) \, \mathrm{d}u = 1 - \int_{\frac{1}{x}}^{1} f(ux) \, \mathrm{d}u.$$

Consider substituting $y = ux$: then, this transforms into

$$x(1 - f(x)) = \int_1^x f(y)\,\mathrm{d}y.$$

Taking the derivative with respect to x, we have $1 - f(x) - xf'(x) = f(x)$, which gives the differential equation $2f(x) + xf'(x) = 1$. Multiplying through by x (integrating factor), we thus have that $x = 2xf(x) + x^2f'(x) = (x^2f(x))'$. Then, $x^2f(x) = \frac{x^2}{2} + C$, or $f(x) = \frac{1}{2} + \frac{C}{x^2}$. Plugging in $f(1) = 1$, we then have that $f(x) = \frac{1}{2} + \frac{1}{2x^2}$. This implies that the final answer is $\frac{1}{2} + \frac{1}{200} = \boxed{\frac{101}{200}}$.

33. Let A, B, and C be points chosen independently and uniformly at random on the unit circle centered at O. Let G be the centroid (center of mass) of triangle $\triangle ABC$. What is the expected value of GO^2?

Answer: $\frac{1}{3}$

Solution: Let $\vec{A}$, $\vec{B}$, and $\vec{C}$ be the coordinates for A, B, and C. Then

$$GO^2 = \left\| \frac{\vec{A} + \vec{B} + \vec{C}}{3} \right\|^2.$$

Since $\vec{A}$, $\vec{B}$, and $\vec{C}$ are independent, the expected values of pairwise dot products are 0. Hence

$$\mathbb{E}[GO^2] = \frac{1}{9}\left(\left\|\vec{A}\right\|^2 + \left\|\vec{B}\right\|^2 + \left\|\vec{C}\right\|^2\right) = \boxed{\frac{1}{3}}.$$

34. Choose 10 numbers $X_1, ..., X_{10}$ independently and uniformly at random from the interval $[0, 1]$. What is the expected value of $\min(X_1, ..., X_{10})$?

Answer: $\frac{1}{11}$

Solution: Write Y as the random variable $\min(X_1, \ldots, X_{10})$; we compute the cumulative probability function $F_Y(y) := \mathbb{P}(Y \leq y)$. Indeed for each $y \in [0, 1]$, we see that

$$F_Y(y) = \mathbb{P}(X_1 \geq y, \ldots, X_{10} \geq y) = 1 - (1 - y)^{10}.$$

Thus, we can compute the probability density function $f_Y(y)$ as $\frac{d}{dy}F_Y(y) = 10(1 - y)^9$, and the expected value can be computed as the integral

$$\int_0^1 yf_Y(y)dy = 10\int_0^1 y(1 - y)^9 = \boxed{\frac{1}{11}}.$$