ICMT — Constellation Round (Division A)

Round Structure

- This is a **team** round. There are **40 problems**, divided into 6 constellations of 6 problems each, plus 4 other problems. Your team will have **90 minutes** total.
- Each of the 6 constellations is associated with a different subject. The subjects are Algebra, Arithmetic (Number Theory), Calculus, Combinatorics, Linear Algebra, and Probability.
- Problems are **NOT** ordered by difficulty.
- Your team may submit an answer to a problem at any time. Each team has **up to 44 submissions**.
- You may resubmit if you get a problem wrong, but only the **LATEST** submission per problem counts.
- During the round, there will be a live scoreboard displaying teams' scores and which problems they have solved (except for Problem 4). The scoreboard **will not be updated in the last 5 minutes**.

Scoring

Your team's score is the sum of the following four categories:

- **Base Points:** Each correct answer receives 15 points. Incorrect answers receive 0 points.
  - One problem (Problem 4) has a different scoring system (which will be revealed during the test), and is worth up to 75 points. It is ineligible for bonus points (explained below).
  - The maximum number of base points is $15 \cdot 39 + 75 = \mathbf{660}$.
- **Speed Bonus:** Bonus points are awarded for solving problems before other teams.
  - The N^{th} team to solve a problem will receive $\max(16 - N, 0)$ bonus points. In other words, the first team to solve a problem will receive 15 bonus points, the second team to solve a problem will receive 14 bonus points, etc. The sixteenth team and later teams will receive no bonus points.
  - We expect roughly 70 teams to be competing.
  - The maximum number of speed bonus points is $15 \cdot 39 = \mathbf{585}$.
- **Depth Bonus:** Bonus points are awarded for solving multiple problems in the same constellation.
  - Within a constellation, $3(N-1)$ bonus points are awarded on the N^{th} correct answer, for a total bonus of $3N(N-1)/2$ depth points per constellation.

Problems solved in constellation	0	1	2	3	4	5	6
Total Depth Bonus	0	0	3	9	18	30	45

  - The maximum number of depth bonus points is $6 \cdot 45 = \mathbf{270}$.
- **Breadth Bonus:** Bonus points are awarded for solving problems in every constellation.
  - For $N > 0$, $18N + 24$ bonus points are awarded upon solving at least N problems in each of the 6 constellations, for a total bonus of $9N(N+1) + 24N$ breadth points.

Problems solved in every constellation	0	1	2	3	4	5	6
Total Breadth Bonus	0	42	102	180	276	390	**522**

The maximum score on this test is $660 + 585 + 270 + 522 = \mathbf{2037}$ points.

Rotating Black Hole ●

1. Let A_3 be the answer to Problem 3. Leanne is playing a game. She begins with a pile of A_3 stones, and then, while there is still a pile with at least three stones remaining, she chooses one such pile and splits it into three piles of positive integral size. If the three pile sizes are a, b, and c, she then earns $ab + ac + bc$ points for this move. If she also earns 1 point per pile of size one and 3 points per pile of size two at the end of the game, what is the maximal total number of points that Leanne can earn?

2. Let A_1 be the answer to Problem 1. Let T_n be the number of ways to split a regular n-gon into $n-2$ triangles such that every triangle is formed by vertices of the n-gon, where rotations and reflections are distinct. Compute the value of
$$\sqrt{\frac{T_{A_1}}{T_{A_1+1}}}.$$

3. Let A_2 be the answer to Problem 2. A simple graph S contains V vertices, and between every pair of vertices, an edge is placed independently with probability $1 - A_2$. What is the minimum value of V such that the expected number of 10-cycles in the graph of S is greater than $10!$? (A 10-cycle is a sequence of vertices $(v_1, v_2, v_3, \ldots, v_{10}, v_1)$ where the vertices $v_1, \ldots, v_{10}$ are all distinct and consecutive vertices are connected with an edge.)

Singular Sun ⊙

4. Submit an integer from 1 to 200, inclusive. Let N be your integer, let T be the total number of teams, let B be the number of teams whose first submission of a valid answer to this problem occurred before your submission, and of those B teams, let A be the number of teams who submitted an answer that differs from N by at most 5. You will receive $\left\lfloor 75 \cdot \frac{N}{200} \cdot \frac{4}{A+4} \cdot \left(\frac{1}{5} + \frac{4}{5} \cdot \frac{B}{T}\right)\right\rfloor$ points for a valid answer, and 0 points for submitting an invalid answer or submitting more than once.

Algebra Aquarius ≈

5. Let (x, y) be a pair of positive integers satisfying $xy^2 - y^2 - x + y = 10$. Compute the sum of all possible values of $x + y$.

6. Let G be a finite abelian group of order 840. What is the maximum possible number of elements of G that have order 14?

7. Compute the order of the group of units of the ring $R = \mathbb{F}_5[x]/(x^{20} - 1)$.

8. Compute the number of complex numbers z such that $|z| = 1$ and
$$z^{340} + z^{140} = -1.$$

9. What is the interval of real values of c such that the equation
$$(x + y)^2 = c(x - 2026)(y + 2026)$$
has exactly one real solution (x, y)?

10. For positive real x, let $f(x) = (x - \sqrt{x}\lfloor\sqrt{x}\rfloor)^2 - \sqrt{x}$. Let c be the smallest positive real number such that $f(c) = (46 - \sqrt{2026})c$. Compute $\lfloor\sqrt{c}\rfloor$.

Arithmetic Aries ♈

11. A polynomial $P(x)$ is called *nice* if $P(1)$ and $P(-1)$ are both divisible by 11, and $P(x)$ is called *small* if all of its coefficients are integers with absolute value at most 5. Calculate the number of nice small polynomials of degree at most 4 (counting $P(x) = 0$).

12. Compute the smallest prime $p > 60$ for which there are three monic quadratic polynomials $q_1(x)$, $q_2(x)$, and $q_3(x)$ with integer coefficients such that all coefficients of the polynomial
$$x^6 + x^3 + 1 - q_1(x)q_2(x)q_3(x)$$
are divisible by p.

13. Compute the number of integer solutions $0 \leq x, y, z < 107$ to the equation
$$x^2 + 2y^2 + 3z^2 \equiv 5 \pmod{107}.$$

14. Compute the largest integer n for which $\varphi(n) \leq 10$.

15. Over all ordered pairs of prime numbers (p, q) such that
$$pq \mid p^{q^2} + q^{2p^2} + 1,$$
compute the sum of all possible values of $p + q$.

16. How many ordered integer triples, (a, b, c), with $1 \leq a, b, c \leq 40$, are there such that a, b, and c form the side lengths of a triangle and $a^2 + ab = c^2$?

Calculus Capricorn ♑

17. Consider sets of the form $A \subseteq [0, 1]$ such that $|A \cap [r, 1]| \leq \frac{1}{r^2}$ for any real number $r \in [0, 1]$. Let S_A be the sum of the elements of such a set A, and let S_n be the maximum value of S_A for all such sets A of size n. Calculate
$$\lim_{n \to \infty} \frac{S_n}{\sqrt{n}}.$$

18. Compute the sum
$$\sum_{n=0}^{\infty} \frac{27(-1)^n}{9n^2 + 15n + 4}.$$

19. A pickup truck is parked at $(20, -21)$ in the (x, y)-plane, and a deer is standing at the origin. Starting from time $t = 0$, the deer runs in the positive y direction at a rate of one unit per second, while the truck always heads directly in the direction of the deer at a rate of two units per second. How many seconds will it take for the truck to reach the deer?

20. There is a unique line that is tangent to the curve $y = x^4 - 8x^3 + 22x^2 - 20x + 26$ at two distinct points and that does not intersect the curve at any other points. Given that the tangent line is of the form $y = ax + b$, compute (a, b).

21. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $f'(x) = f(x) - x$ for all real x and $f(x) > 0$ for all $x > -2$. What is the minimum possible value of $f(0)$?

22. For a positive integer $n \geq 2$, let $S_n = \left\{\frac{1}{k} - \frac{1}{n} : 1 \leq k \leq n-1, k \in \mathbb{Z}\right\}$, and let g_n be the geometric mean of S_n. Compute
$$\lim_{n \to \infty} (1 - g_n)^n.$$

Combinatorics Cancer ♋

23. There are six people living on the same floor of a dorm, where the rooms are numbered $1, 2, \ldots, 6$ (each student lives in a different room). Each pair of residents is either friends or enemies. For every triple of integers (i, j, k) with $1 \leq i < j < k \leq 6$, the following are true:

    - If the resident in room j is friends with the residents in both room i and room k, then each pair among the three are friends.
    - If the resident in room j is enemies with the residents in both room i and room k, then each pair among the three are enemies.

    How many possible friendship configurations are there among these six people?

24. Michelle is playing a game with 60 slips of paper, each numbered with a distinct positive integer from 1 to 60. She first lays the slips down in a random order in a line. She can then take two **consecutive** slips and swap them. What is the least number of swaps necessary to guarantee that any initial arrangement can be changed to one where all even numbers are next to at least one other even number and all odd numbers are next to at least one other odd number?

25. Rohit is traveling on the (x, y)-plane, starting at $(0, 0)$, and can only move in the following two ways:

$$(x, y) \mapsto (x + 3, y + 2), \qquad (x, y) \mapsto (x - 3, y - 1).$$

    How many possible sequences of moves allow Rohit to reach the point $(6, 7)$?

26. The 20-dimensional Boolean hypercube H_{20} is a graph G on vertex set $V = \{0, 1\}^{20}$, where vertex $x = (x_1, \ldots, x_{20})$ has an edge to $y = (y_1, \ldots, y_{20})$, if, for exactly a single $i \in \{1, 2, \ldots, 20\}$, $x_i \neq y_i$. Now, vertices are removed from the hypercube H_{20}, disconnecting it, until one connected component has 35 vertices. What is the minimum number of vertices that could have been removed?

27. Ten people are attending a paintball tournament and will be assigned to two opposing teams of 5: Red Team and Blue Team. (The teams are labeled, so swapping Red and Blue counts as a different team selection.) However, some people are enemies with each other and do not want to be on the same team. For a given set of pairs of enemies E, let $f(E)$ be the number of "valid team selections" with no enemy pairs on the same team. What is the sum of $f(E)$ over all possible sets of enemy pairs?

28. Let $A_1, A_2, \ldots, A_k \subset \mathbb{R}$ be sets of size at most 2030 such that $\bigcap_{i=1}^{k} A_i = \varnothing$. Suppose there exists an integer $t \geq 1$ such that $|A_i \cap A_j| = t$ for all $1 \leq i < j \leq k$. Calculate the maximum possible value of k.

Linear Libra ♎

29. Let $V = \mathbb{R}[X]_{\leq 2025}$ be the vector space of real polynomials of degree at most 2025. Consider the derivative map $f : V \to V$, defined by

$$f\left(\sum_{i=0}^{2025} a_i X^i\right) = \sum_{i=1}^{2025} i a_i X^{i-1},$$

    where each $a_i \in \mathbb{R}$. Calculate the eigenvalue of f with the largest magnitude.

30. Let $X = \mathbb{C}^{100\times 100}$ and let $T : X \to \mathbb{C}$ be a linear map such that

    - $T(AB) = T(BA)$ for every $A, B \in X$, and
    - if $C = (c_{jk}) \in X$ is the diagonal matrix with $c_{nn} = n$ for $1 \leq n \leq 100$, then $T(C) = 101$.

    Compute $T(M)$, where $M = (m_{jk}) \in X$ is the matrix where $m_{jk} = 1$ if $j + k$ is even and 0 otherwise.

31. Let R be the ring $\mathbb{F}_3[t]/(t^4)$. Let N be the number of 3×3 matrices, $M \in R^{3\times 3}$, such that $\det(M) = 1$ (i.e. the determinant of M is the constant polynomial $1 \in R$). Calculate the number of divisors of N (including 1 and N).

32. Compute the largest possible integer m such that there exists $2m$ unit vectors $v_1, v_2, \ldots, v_m$ and $w_1, w_2, \ldots, w_m$ in $\mathbb{R}^{100}$ with the property that for all $1 \le i < j \le m$,
    - $v_i \cdot v_j = -1/50$,
    - $w_i \cdot w_j = -1/50$,
    - $v_i \cdot w_j = -1/100$, and
    - $v_i \cdot w_i = 0$.

33. Calculate the dimension of the vector space S of all 4×4 real symmetric matrices that commute with
$$X = \begin{pmatrix} 1 & -1 & -3 & -1 \\ -1 & 1 & -1 & -3 \\ -3 & -1 & 1 & -1 \\ -1 & -3 & -1 & 1 \end{pmatrix}.$$

34. Compute the determinant of the matrix
$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 5 & 4 & 3 \\ 1 & 4 & 25 & 16 & 9 \\ 1 & 8 & 125 & 64 & 27 \\ 1 & 16 & 625 & 256 & 0 \end{pmatrix}.$$

Probability Pisces ♓

35. Cat and David are playing a game. They begin with the number 10 written on a whiteboard. The two then take turns, starting with Cat. On a player's turn, the player chooses a number x uniformly at random from the interval $[0, 1]$, independently of all previous choices, and multiplies the number on the whiteboard by x. They then replace the number on the whiteboard with this product. The game continues until the number on the whiteboard is at most 1. What is the probability that Cat is the last person to take a turn?

36. Kaity is playing a number game. She writes the number 324000 on a whiteboard, and at each step she chooses one of the positive divisors of the most recently written number uniformly at random and writes it on the whiteboard. She repeats this process until she writes the number 1. What is the expected value of the natural logarithm of the product of all the numbers written on the whiteboard, including the initial number 324000?

37. Let X be a real random variable such that $\mathbb{E}[X^{2k}] = (2k + 1)!$ for any positive integer k. Compute the minimum possible value of $\mathbb{P}(X \le 2026)$.

38. Let A, B, and C be points chosen independently and uniformly at random on the unit circle centered at O. Let G be the centroid (center of mass) of triangle $\triangle ABC$. What is the expected value of GO^2?

39. Let x, y, z be 3 random elements chosen independently and uniformly from $\{-1, 1\}^n \subset \mathbb{R}^n$. Let p_n be the probability that the triangle formed by x, y, z (as points in $\mathbb{R}^n$) is acute. Calculate
$$\lim_{n\to\infty} \frac{\ln\big(8^n(1 - p_n)\big)}{n}.$$

40. A cube of side length 2026 is painted on the exterior and is then cut into 2026^3 unit cubes. A unit cube is then chosen uniformly at random and is rolled twice independently. Given that the top face on the first roll is painted, what is the probability that the top face on the second roll is also painted?