ICMT — Constellation Round Solutions (Division A)

Rotating Black Hole ●

1. Let A_3 be the answer to Problem 3. Leanne is playing a game. She begins with a pile of A_3 stones, and then, while there is still a pile with at least three stones remaining, she chooses one such pile and splits it into three piles of positive integral size. If the three pile sizes are $a, b,$ and c, she then earns $ab + ac + bc$ points for this move. If she also earns 1 point per pile of size one and 3 points per pile of size two at the end of the game, what is the maximal total number of points that Leanne can earn?

   Answer: 120

   Solution: We will show that by induction, Leanne's score will always be $\frac{A_3(A_3+1)}{2}$.

   Base case: Leanne earns $\frac{1(2)}{2} = 1$ points starting with 1 stone and $\frac{2(3)}{2} = 3$ points starting with 2 stones.

   Inductive Step: For any positive integer $N \geq 3$, suppose that that for all $n \leq N$, starting with a pile of n stones, any sequence of moves will lead to Leanne earning $\frac{n(n+1)}{2}$ points. We show that starting with $N + 1$ stones will lead to Leanne earning $\frac{(N+1)(N+2)}{2}$ points.

   Leanne must split the pile into sizes a, b, and c, with $1 \leq a, b, c, \leq N - 2$ and $a + b + c = N + 1$. This will earn her $ab + bc + ca$ points, and by the inductive hypothesis, she in total will earn

$$\frac{a(a+1)}{2} + \frac{b(b+1)}{2} + \frac{c(c+1)}{2} + ab + bc + ca = \frac{(a+b+c)(a+b+c+1)}{2} = \frac{(N+1)(N+2)}{2}$$

   points, as desired.

   This completes the induction. Using that $A_3 = 15$, we have that our answer for this problem is $\boxed{120}$.

2. Let A_1 be the answer to Problem 1. Let T_n be the number of ways to split a regular n-gon into $n - 2$ triangles such that every triangle is formed by vertices of the n-gon, where rotations and reflections are distinct. Compute the value of

$$\sqrt{\frac{T_{A_1}}{T_{A_1+1}}}.$$

   Answer: $2\sqrt{\frac{5}{79}}$

   Solution:

   One can show that $T_n = C_{n-2}$, where C_n represents the Catalan numbers, by a variety of combinatorial techniques. Thus

$$\sqrt{\frac{T_{A_1}}{T_{A_1+1}}} = \sqrt{\frac{\binom{2(A_1-2)}{A_1-2}}{A_1 - 1} \cdot \frac{A_1}{\binom{2(A_1-1)}{A_1-1}}} = \sqrt{\frac{A_1}{2(2A_1 - 3)}}.$$

   Taking in the answer $A_1 = 120$ gives $\boxed{2\sqrt{\frac{5}{79}}}$ as the answer.

3. Let A_2 be the answer to Problem 2. A simple graph S contains V vertices, and between every pair of vertices, an edge is placed independently with probability $1 - A_2$. What is the minimum value of V such that the expected number of 10-cycles in the graph of S is greater than 10!? (A 10-cycle is a sequence of vertices $(v_1, v_2, v_3, \ldots, v_{10}, v_1)$ where the vertices $v_1, \ldots, v_{10}$ are all distinct and consecutive vertices are connected with an edge.)

   Answer: 15

Solution: By linearity of expectation, the expected value of S is $\binom{V}{10}10!(1-A_2)^{10}$.

Once all of the problems have been solved in terms of each other, problem 3 is the natural place to break in to get numerical answers. The answer to problem 3 must be at least 10, so the answer to problem 1 will be large (at least 50), and so we conclude that the answer to problem 2 is close to $1/2$.

Note that the answer to problem 2 will always be greater than $1/2$, so we conclude that the expected value of S satisfies

$$\binom{V}{10}10!(1-A_2)^{10} \le \binom{V}{10}10!2^{-10}.$$

In particular, if $\binom{V}{10} \le 2^{10}$, we then have that the expected number of 10-cycles in the graph is at most $10!$. Since $\binom{14}{10} < 2^{10}$ and $\binom{15}{10} > 2^{10}$, we conclude that the answer to this problem must be at least 15. One can in fact verify that $\boxed{15}$ is the only possible answer; if $A_3 \ge 16$, one can show that $A_3 - 1$ is also a valid value of V, and hence A_3 is not the minimum possible value of V.

Singular Sun ⊙

4. Submit an integer from 1 to 200, inclusive. Let N be your integer, let T be the total number of teams, let B be the number of teams whose first submission of a valid answer to this problem occurred before your submission, and of those B teams, let A be the number of teams who submitted an answer that differs from N by at most 5. You will receive $\left\lfloor 75 \cdot \frac{N}{200} \cdot \frac{4}{A+4} \cdot \left(\frac{1}{5} + \frac{4}{5} \cdot \frac{B}{T}\right)\right\rfloor$ points for a valid answer, and 0 points for submitting an invalid answer or submitting more than once.

   Solution: N/A

Algebra Aquarius ≈

5. Let (x, y) be a pair of positive integers satisfying $xy^2 - y^2 - x + y = 10$. Compute the sum of all possible values of $x + y$.

   Answer: 23

   Solution: We can write that

   $$x = \frac{10 - y + y^2}{y^2 - 1}.$$

   Since $x \ge 1$, we must have that $y \le 11$. Hence $1 \le y \le 11$, so we can check all cases. There are 3 solutions: $(1, 11)$, $(2, 3)$, and $(4, 2)$, which sum to $\boxed{23}$.

6. Let G be a finite abelian group of order 840. What is the maximum possible number of elements of G that have order 14?

   Answer: 42

   Solution: Write $840 = 2^3 \cdot 3 \cdot 5 \cdot 7$. By the primary decomposition for finite abelian groups, we may write

   $$G \cong G_{(2)} \times G_{(3)} \times G_{(5)} \times G_{(7)},$$

   where $|G_{(2)}| = 8$, $|G_{(3)}| = 3$, $|G_{(5)}| = 5$, and $|G_{(7)}| = 7$.

   An element $g = (g_2, g_3, g_5, g_7) \in G$ has order $14 = 2 \cdot 7$ if and only if $g_3 = e$, $g_5 = e$, $\mathrm{ord}(g_2) = 2$, and $\mathrm{ord}(g_7) = 7$. Hence the number of elements of order 14 in G is

   $$\big(\#\{g_2 \in G_{(2)} : \mathrm{ord}(g_2) = 2\}\big) \cdot \big(\#\{g_7 \in G_{(7)} : \mathrm{ord}(g_7) = 7\}\big).$$

   Since $G_{(7)} \cong \mathbb{Z}_7$, there are 6 elements of order 7 in $G_{(7)}$. To maximize the number of elements of order 2 in $G_{(2)}$ (with $|G_{(2)}| = 8$), we take $G_{(2)} \cong (\mathbb{Z}_2)^3$, which has exactly 7 elements of order 2. Therefore the maximum possible number of elements of order 14 in G is $7 \cdot 6 = \boxed{42}$.

7. Compute the order of the group of units of the ring $R = \mathbb{F}_5[x]/(x^{20}-1)$.

Answer: $2^8 \cdot 5^{16}$

Solution: We have that

$$\mathbb{F}_5[x]/(x^{20}-1) \cong \mathbb{F}_5[x]/(x^4-1)^5 \cong \mathbb{F}_5[x]/((x-1)(x-2)(x-3)(x-4))^5$$

and by CRT this is isomorphic to 4 copies of $\mathbb{F}_5[x]/(x^5)$. The group of units of $\mathbb{F}_5[x]/(x^5)$ are polynomials of the form $a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ with a_0 a unit in $\mathbb{F}_5$ and $a_1, a_2, a_3, a_4 \in \mathbb{F}_5$, so we get $4 \cdot 5^4 = 2500$ units in $\mathbb{F}_5[x]/(x^5)$, and in total the order of the group of units is $2500^4 = \boxed{2^8 \cdot 5^{16}}$.

8. Compute the number of complex numbers z such that $|z| = 1$ and

$$z^{340} + z^{140} = -1.$$

Answer: 40

Solution: Let $w = z^{20}$, so that we want to find complex numbers satisfying $w^{17} + w^7 + 1 = 0$. We need $w = e^{i\theta}$ since the magnitude of w is $|z|^{20} = 1$. Then, in order for $e^{i\theta} + e^{i\phi} = -1$, we need $\theta, \phi = \frac{2\pi}{3}, \frac{4\pi}{3}$ in some order (i.e. $e^{i\theta}, e^{i\phi} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$.) This gives $7\theta, 17\theta = \frac{2\pi}{3} + 2\pi k, \frac{4\pi}{3} + 2\pi l$ in some order. We get $3\theta = \pm 2\pi(l-2k)$, which either fixes $\theta = \frac{2\pi}{3} + 2\pi x$ or $\theta = \frac{4\pi}{3} + 2\pi x$. Thus, our two solutions for w are $w = e^{2\pi i/3}, e^{4\pi i/3}$. Then, $z^{20} = w$ has 20 unique solutions for each of those cases, giving a total of $2 \cdot 20 = \boxed{40}$ possible values of z.

9. What is the interval of real values of c such that the equation

$$(x+y)^2 = c(x-2026)(y+2026)$$

has exactly one real solution (x, y)?

Answer: $(0, 4)$

Solution: Making the substitution $u = x - 2026$ and $v = y + 2026$, this is equivalent to $(u+v)^2 = cuv$, or $u^2 + (2-c)uv + v^2 = 0$.

We will always have the real solution $(u, v) = (0, 0)$. When $v \neq 0$, we may also gain the real solutions $\frac{u}{v} = r$, if r is a real root of the quadratic

$$r^2 + (2-c)r + 1 = 0.$$

We only want a single real solution, so we want there to be no such real r. This occurs precisely when $(2-c)^2 - 4 < 0$, or when $c \in \boxed{(0, 4)}$.

10. For positive real x, let $f(x) = (x - \sqrt{x}\lfloor\sqrt{x}\rfloor)^2 - \sqrt{x}$. Let c be the smallest positive real number such that $f(c) = (46 - \sqrt{2026})c$. Compute $\lfloor\sqrt{c}\rfloor$.

Answer: 90

Solution: For nonnegative integer k, suppose $k \leq \sqrt{x} < k+1$, so that $\lfloor\sqrt{x}\rfloor = k$. Then we define $f_k(x) = (x - k\sqrt{x})^2 - \sqrt{x}$, which equals $f(x)$ on the interval $[k^2, (k+1)^2)$. We are attempting to find the minimum k such that there is c in the domain of f_k that satisfies $\frac{f_k(c)}{c} = 46 - \sqrt{2026}$, so we consider $g_k(x) := \frac{f_k(x)}{x}$:

$$g_k(x) = (\sqrt{x} - k)^2 - \frac{1}{\sqrt{x}}$$

Taking a derivative gives

$$g_k'(x) = 2(\sqrt{x} - k)\frac{1}{2\sqrt{x}} + \frac{1}{2x\sqrt{x}} = 1 - \frac{k}{\sqrt{x}} + \frac{1}{2x\sqrt{x}}$$

which is always positive on the interval $[k^2,(k+1)^2)$, meaning $g_k(x)$ is an increasing function. Therefore, we want to find the minimal k where $g_k((k+1)^2) \geq 46-\sqrt{2026}$, as this is the supremum of the range of g_k. We have $g_k((k+1)^2) = 1-\frac{1}{k+1}$, so we need to bound $46-\sqrt{2026}$ in an interval of the form $(1-\frac{1}{k}, 1-\frac{1}{k+1}]$.

Alternatively, we can bound $1-(46-\sqrt{2026}) = \sqrt{2026}-45 = \sqrt{2026}-\sqrt{2025}$ in an interval of the form $[\frac{1}{k+1}, \frac{1}{k})$. Note that $\sqrt{x}$ is concave down, so that the tangent line is above the function. Therefore the tangent to $x = 2025$ with slope $\frac{1}{2\sqrt{2025}} = \frac{1}{90}$ is above the function at $x = 2026$, meaning $\sqrt{2026} < \sqrt{2025}+\frac{1}{90}$. Also, the secant line from $x = 45^2$ to $x = 46^2$ is below the function on the interval $[45^2, 46^2)$, so $\sqrt{2026} > \sqrt{2025}+\frac{1}{91}\cdot(\sqrt{46^2}-\sqrt{45^2}{=}\sqrt{2025}+\frac{1}{91}$. Thus, $46-\sqrt{2026} \in (1-\frac{1}{90}, 1-\frac{1}{91}]$ and our answer is $k = \boxed{90}$.

Arithmetic Aries ♈

11. A polynomial $P(x)$ is called *nice* if $P(1)$ and $P(-1)$ are both divisible by 11, and $P(x)$ is called *small* if all of its coefficients are integers with absolute value at most 5. Calculate the number of nice small polynomials of degree at most 4 (counting $P(x) = 0$).

    Answer: 1331

    Solution: Notice that the set of nice polynomials can be identified with $\mathbb{F}_{11}^5$. Furthermore, the set of polynomials which satisfy $P(1) = P(-1) = 0$ form a co-dimension 2 subspace of this vector space. Thus the number of nice polynomials is $11^{5-2} = \boxed{1331}$.

12. Compute the smallest prime $p > 60$ for which there are three monic quadratic polynomials $q_1(x)$, $q_2(x)$, and $q_3(x)$ with integer coefficients such that all coefficients of the polynomial

    $$x^6 + x^3 + 1 - q_1(x)q_2(x)q_3(x)$$

    are divisible by p.

    Answer: 71

    Solution: Let ζ be a primitive ninth root of unity, and define $\Phi(x) = x^6+x^3+1$. Then the factorization

    $$\Phi(x) = \prod_{\substack{1\leq i\leq 9 \\ \gcd(i,9)=1}} (x-\zeta^i)$$

    holds in $\mathbb{Z}[\zeta][x]$, so it will hold in $\overline{\mathbb{F}_p}$ by taking a reduction. We now have the following cases.

    - If $p \equiv 1 \pmod 9$, then $\zeta^p = \zeta$, so $\zeta \in \mathbb{F}_p$, so $\Phi(x) \pmod p$ factors into linear factors.
    - If $p^2 \equiv 1 \pmod 9$ but $p \not\equiv 1$, then $\zeta^{p^2} = \zeta$ but $\zeta^p \neq \zeta$, so $\zeta^p = \zeta^{-1}$ and $\zeta \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$. In particular, Φ cannot factor into linear factors, but it does factor into the quadratic polynomials

      $$\left(x-\zeta^i\right)\left(x-\zeta^{pi}\right).$$

    - If $p^3 \equiv 1 \pmod 9$ but $p \not\equiv 1$, then the same argument as in the previous point shows that Φ does not factor into linear factors, but it does factor into cubics.
    - Lastly, if $p \pmod 9$ is a generator, then $\mathbb{F}_p[\zeta] = \mathbb{F}_{p^6}$, so $\Phi \pmod p$ is irreducible.

    Thus, we see that the problem asks us to hunt for primes p for which $p \equiv \pm 1 \pmod 9$. The first such prime exceeding 60 is $\boxed{71}$.

13. Compute the number of integer solutions $0 \leq x, y, z < 107$ to the equation

    $$x^2 + 2y^2 + 3z^2 \equiv 5 \pmod{107}.$$

Answer: 11342

Solution: Let $p = 107$. Note that the number of solutions to $x^2 \equiv a \pmod p$ is precisely $1 + \left(\frac{a}{p}\right)$. Letting $a = x^2$, $b = 2y^2$, and $c = 3z^2$, we have that the number of solutions to the above equation can be written as

$$\sum_{\substack{a+b+c=5 \\ 0 \le a,b,c<p}} \left(1 + \left(\frac{a}{p}\right)\right)\left(1 + \left(\frac{2b}{p}\right)\right)\left(1 + \left(\frac{3c}{p}\right)\right).$$

It now remains to compute this sum. We split the sum into three cases based on the number of Legendre symbols multiplied together:

Case 1: No Legendre symbols; i.e. this is the sum

$$\sum_{\substack{a+b+c=5 \\ 0 \le a,b,c<p}} 1.$$

For any choice of a and b, there is precisely one choice of c in the sum. Hence this sum evaluates to p^2.

Case 1: One Legendre symbol; i.e. this is the sum

$$\sum_{\substack{a+b+c=5 \\ 0 \le a,b,c<p}} \left(\frac{a}{p}\right),$$

and similarly for $2b$ and $3c$. Summing over all a gives 0. The same can be done for $2b$ and $3c$ (since $p > 3$), so all of the sums evaluates to 0.

Case 2: Two Legendre symbols; i.e. this is the sum

$$\sum_{\substack{a+b+c=5 \\ 0 \le a,b,c<p}} \left(\frac{a}{p}\right)\left(\frac{2b}{p}\right),$$

and similarly for the other pairs. For any choice of a and b, here is precisely one choice of c in the sum, hence We can sum over all a and b to get 0. The same applies to the other pairs, so all of the sums evaluate to 0.

Case 2: Three Legendre symbols; i.e. this is the sum

$$\sum_{\substack{a+b+c=5 \\ 0 \le a,b,c<p}} \left(\frac{a}{p}\right)\left(\frac{2b}{p}\right)\left(\frac{3c}{p}\right).$$

We can rewrite the sum in the form

$$\begin{aligned}
\sum_{\substack{a+b+c=5 \\ 0 \le a,b,c<p}} \left(\frac{a}{p}\right)\left(\frac{2b}{p}\right)\left(\frac{3c}{p}\right) &= \sum_{0 \le a,b<p} \left(\frac{a}{p}\right)\left(\frac{2b}{p}\right)\left(\frac{3(5-a-b)}{p}\right) \\
&= \sum_{0<a,b<p} \left(\frac{a}{p}\right)^2\left(\frac{2}{p}\right)\left(\frac{b}{p}\right)^2\left(\frac{15\overline{a}\overline{b} - 3\overline{b} - 3\overline{a}}{p}\right) \\
&= \sum_{0<a,b<p} \left(\frac{2}{p}\right)\left(\frac{15\overline{a}\overline{b} - 3\overline{b} - 3\overline{a}}{p}\right),
\end{aligned}$$

where in the second equality we note that the sum is 0 whenever a or b is 0, and $\overline{a}$ and $\overline{b}$ denote the multiplicative inverses of a and b modulo p, respectively. We note that for each fixed choice of $0 < a < p$ with $a \neq 5$, as we vary over $0 < b < p$, $15\overline{a}\overline{b} - 3\overline{b} - 3\overline{a}$ varies over all equivalence classes mod p except for $-3\overline{a}$. However, when $a = 5$, the inner summand becomes $\left(\frac{-3\cdot\overline{5}}{p}\right)$ for all b. Hence our sum can be written as

$$\sum_{0<a,b<p}\left(\frac{2}{p}\right)\left(\frac{15\overline{a}\overline{b}-3\overline{b}-3\overline{a}}{p}\right) = \left(\frac{2}{p}\right)\left(\sum_{\substack{0<a<p\\a\neq 5}} -\left(\frac{-3\overline{a}}{p}\right) + \sum_{0<b<p}\left(\frac{-3\cdot\overline{5}}{p}\right)\right).$$

Finally, varying over all $0 < a < p$, $a \neq 5$, $-3\overline{a}$ varies over all equivalence classes mod p except for 0 and $-3\cdot\overline{5}$. Thus our sum evaluates as

$$\left(\frac{2}{p}\right)\left(\left(\frac{-3\cdot\overline{5}}{p}\right) + (p-1)\left(\frac{-3\cdot\overline{5}}{p}\right)\right) = p\left(\frac{-1}{p}\right)\left(\frac{2}{p}\right)\left(\frac{3}{p}\right)\left(\frac{5}{p}\right).$$

Hence, the total number of solutions is

$$p^2 + p\left(\frac{-1}{p}\right)\left(\frac{2}{p}\right)\left(\frac{3}{p}\right)\left(\frac{5}{p}\right).$$

Evaluating this in the case of $p = 107$, we get that the number of solutions is

$$107^2 + (107)(-1)(-1)(1)(-1) = \boxed{11342}.$$

14. Compute the largest integer n for which $\varphi(n) \leq 10$.

Answer: 30

Solution: To begin, note that n cannot admit a prime factor larger than 11: the surjection $(\mathbb{Z}/n\mathbb{Z})^\times \to (\mathbb{Z}/p\mathbb{Z})^\times$ reveals that $(\mathbb{Z}/n\mathbb{Z})^\times$ has at least $p - 1 > 10$ elements! Thus,

$$\frac{\varphi(n)}{n} \geq \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{4}{5}\cdot\frac{6}{7}\cdot\frac{10}{11} = \frac{16}{77} > \frac{1}{5},$$

so $n < 50$, and similarly $n < 25$ if n is odd. Some quick casework reveals that $n = \boxed{30}$ is the largest possible value.

15. Over all ordered pairs of prime numbers (p, q) such that

$$pq \mid p^{q^2} + q^{2p^2} + 1,$$

compute the sum of all possible values of $p + q$.

Answer: 20

Solution:

Taking mod q, we have that $p^{q^2} \equiv -1 \pmod{q}$. If q is 2, we conclude that p must be odd. If q is an odd prime, the order of p mod q must divide $2q^2$ and must not divide q^2. Moveover, by Fermat's Little Theorem, the order must be at most $q - 1$. Hence the order of p mod q is 2. Thus, either $q = 2$ and p is odd, or q is odd and $q \mid p + 1$.

Taking mod p, we have that $q^{2p^2} \equiv -1 \pmod{p}$. If p is 2, we conclude that q must be odd. If p is an odd prime, the order of q mod p must divide $4p^2$ and must not divide $2p^2$, and must be at most $p - 1$. Hence the order of q mod p is 4. Thus, either $p = 2$ and q is odd, or p is odd and $p \mid q^2 + 1$.

We now evaluate all possible cases:

Case 1: $p = 2$. Then q is an odd prime with $q \mid p + 1$, hence $q = 3$.

Case 2: $q = 2$. Then p is an odd prime with $p \mid q^2 + 1$, hence $p = 5$.

Case 3: p and q are odd primes. Then $q \mid p + 1$ and $p \mid q^2 + 1$. In particular, we can write $p + 1 = kq$, for some positive integer $k < p + 1$. Since $q^2 \equiv -1 \pmod{p}$, we get that $k \equiv -q \pmod{p}$, and hence $k = p - q$. Thus, we are looking for solutions to $p + 1 = q(p - q)$. Solving gives that $p = \frac{q^2+1}{q-1}$; since $\gcd(q^2 + 1, q - 1) = 2$, this is only possible if $q = 3$ and $p = 5$.

This exhausts all possibilities, and thus the sum of all possible $p + q$ is $5 + 7 + 8 = \boxed{20}$.

16. How many ordered integer triples, (a, b, c), with $1 \leq a, b, c \leq 40$, are there such that $a, b,$ and c form the side lengths of a triangle and $a^2 + ab = c^2$?

Answer: 17

Solution: The difficulty in this problem lies in parameterizing and efficiently iterating through all possible solutions. We outline a strategy below.

For now, we assume that a, b are relatively prime. Then we can parameterize all solutions in the form $a = x^2$, $b = y^2 - x^2$, and $c^2 = (xy)^2$, with $\gcd(x, y) = 1$. Since $a, b,$ and c form a triangle, the triangle inequality also gives relation $x < y < 2x$.

We now can iterate through all possible values of xy up to 40, with $\gcd(x, y) = 1$ and $x < y < 2x$. To get the non relatively prime solutions, we can scale x and y by a constant multiple. Iterating all possible solutions gives $\boxed{17}$ solutions: (4, 5, 6), (8, 10, 12), (9, 7, 12), (9, 16, 15), (12, 15, 18), (16, 9, 20), (16, 20, 24), (18, 14, 24), (16, 33, 28), (18, 32, 30), (20, 25, 30), (25, 11, 30), (25, 24, 35), (24, 30, 36), (27, 21, 36), (25, 39, 40), (32, 18, 40).

Calculus Capricorn ♑

17. Consider sets of the form $A \subseteq [0, 1]$ such that $|A \cap [r, 1]| \leq \frac{1}{r^2}$ for any real number $r \in [0, 1]$. Let S_A be the sum of the elements of such a set A, and let S_n be the maximum value of S_A for all such sets A of size n. Calculate

$$\lim_{n\to\infty} \frac{S_n}{\sqrt{n}}.$$

Answer: 2

Solution: Note that

$$\begin{aligned} S_A &= \sum_{a\in A} a = \sum_{a\in A} \int_0^a dx = \int_0^1 \#\{a \in A : a \geq x\}\, dx \\ &\leq \int_0^1 \min\left(n, \frac{1}{x^2}\right) dx = \int_0^{1/\sqrt{n}} n\, dx + \int_{1/\sqrt{n}}^1 dx/x^2 \\ &= 2\sqrt{n} - 1, \end{aligned}$$

giving an upper bound of $\boxed{2}$ for the limit. Tightness (in the limit) is achieved by the sets

$$A_n := \{1/\sqrt{k} : k \leq n\}$$

.

18. Compute the sum

$$\sum_{n=0}^{\infty} \frac{27(-1)^n}{9n^2 + 15n + 4}.$$

Answer: $2\sqrt{3}\pi - 9 + 6\ln(2)$

Solution: We can rewrite the sum in the form

$$\begin{aligned} 9\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{3n+1} - \frac{(-1)^n}{3n+4}\right) =&9\sum_{n=0}^{\infty}(-1)^n\left(\int_0^1 x^{3n}\, dx - \int_0^1 x^{3n+3} dx\right) \\ =&9\int_0^1 \frac{1-x^3}{1+x^3}\, dx. \end{aligned}$$

The integral can now be evaluated by partial fractions; in particular,

$$\int \frac{1-x^3}{1+x^3}\,dx = -x - \frac{1}{3}\log\bigl(x^2-x+1\bigr) + \frac{2}{3}\log(x+1) + \frac{2\sqrt{3}}{3}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C.$$

Evaluating at the bounds gives the answer $\boxed{2\sqrt{3}\pi - 9 + 6\ln(2)}$.

19. A pickup truck is parked at $(20,-21)$ in the (x,y)-plane, and a deer is standing at the origin. Starting from time $t=0$, the deer runs in the positive y direction at a rate of one unit per second, while the truck always heads directly in the direction of the deer at a rate of two units per second. How many seconds will it take for the truck to reach the deer?

Answer: $\frac{79}{3}$

Solution: (Based on this Stackexchange post.) Let $(x(t),y(t))$ describe the path of the pickup truck; we wish to find the time t such that $x(t)=0$. In particular, $x(0)=20$ and $y(0)=-21$. Since the truck is always traveling toward the left, we also can express y as a function of x, hence $\frac{\mathrm{d}y}{\mathrm{d}x}$ is well-defined.

We start by translating the information that truck is always headed directly at the deer. At time t, the deer will be at $(0,t)$, and the truck will be at (x,y), so we must have that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y-t}{x}.$$

We may solve for t to get that

$$t = y - x\frac{\mathrm{d}y}{\mathrm{d}x}$$

Next, since the truck always travels at a rate of two units per second, after time t it will have traveled $2t$ units. By the arc length formula, we must have that

$$2t = \int_x^{20} \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x\,.$$

Thus, we have that

$$2\left(y - x\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \int_x^{20} \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x\,.$$

We now take the derivative of both sides with respect to x. Doing so gives

$$2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}.$$

Let $u = \frac{\mathrm{d}y}{\mathrm{d}x}$. We then have the differential equation

$$2x\frac{\mathrm{d}u}{\mathrm{d}x} = \sqrt{1+u^2},$$

and separation of variables gives

$$\sinh^{-1}(u) = \frac{1}{2}\ln(x) + C$$

for some constant C. Using that $x(0)=20$ and $u(0)=\frac{-21}{20}$, we have that

$$C = \sinh^{-1}\left(-\frac{21}{20}\right) - \frac{1}{2}\ln(20) = -\frac{3}{2}\ln 5$$

Solving for u then gives

$$u = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2 \cdot 5^{3/2}}\sqrt{x} - \frac{5^{3/2}}{2\sqrt{x}},$$

and so

$$y = \frac{1}{3 \cdot 5^{3/2}} x^{3/2} - 5^{3/2}\sqrt{x} + C$$

for a different constant C. Again, since $x(0) = 20$ and $y(0) = -21$, we have that

$$C = -21 - \left(\frac{1}{3 \cdot 5^{3/2}} 20^{3/2} - 5^{3/2}\sqrt{20}\right) = \frac{79}{3},$$

and so

$$y = \frac{2}{3} 5^{-3/2} x^{3/2} - \frac{5^{3/2}}{2}\sqrt{x} + \frac{79}{3}.$$

Finally, the collision between the deer and pickup truck occurs when $x = x(t) = 0$. When this occurs, we have that $y(t) = \frac{79}{3}$. Since the deer starts at $(0, 0)$ and travels upward at rate of one unit per second, we conclude that the final collision occurred at $\boxed{\frac{79}{3}}$ seconds.

20. There is a unique line that is tangent to the curve $y = x^4 - 8x^3 + 22x^2 - 20x + 26$ at two distinct points and that does not intersect the curve at any other points. Given that the tangent line is of the form $y = ax + b$, compute (a, b).

Answer: $(4, 17)$

Solution: Let $f(x)$ be the polynomial. Then for some real numbers a and b, we want $f(x) - ax - b$ to have two different double roots s and t. By Vieta's formulas, we want $2s+2t = 8$ and $s^2+t^2+4st = 22$. Solving for s and t gives the solutions of 1 and 3. Then

$$ax + b = (x^4 - 8x^3 + 22x^2 - 20x + 26) - ((x-1)(x-3))^2 = 4x + 17,$$

giving the answer $\boxed{(4, 17)}$.

21. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $f'(x) = f(x) - x$ for all real x and $f(x) > 0$ for all $x > -2$. What is the minimum possible value of $f(0)$?

Answer: $e^2 + 1$

Solution: To solve $f(x) - f'(x) = x$, we find some solution to $f(x) - f'(x) = x$ as a polynomial, and then we can add Ce^x with some constant C as a solution to $f(x) - f'(x) = 0$ that won't change the relation. Plugging in $f(x) = ax + b$, we get $f(x) - f'(x) = ax + (b - a) = x$, meaning $a = b = 1$, so $f(x) = x + 1 + Ce^x$.

Then $f(0) = C + 1$, so minimizing $f(0)$ corresponds to minimizing C. First, we note that C must be positive, as we need that $f(x)$ is positive as $x \to \infty$. Then minimizing C corresponds to minimizing the value of $f(-2)$, since e^x is always positive. Since $f(x) > 0$ for all $x > -2$, the best we can do is setting $f(-2) = 0$. Therefore $f(-2) = 0 = -1 + Ce^{-2}$ giving $C = e^2$, and $f(0) = \boxed{e^2 + 1}$.

22. For a positive integer $n \geq 2$, let $S_n = \left\{\frac{1}{k} - \frac{1}{n} : 1 \leq k \leq n-1, k \in \mathbb{Z}\right\}$, and let g_n be the geometric mean of S_n. Compute

$$\lim_{n\to\infty} (1 - g_n)^n.$$

Answer: $1/e$

Solution: Observe that

$$g_n^{n-1} = \prod_{i=1}^{n-1}\left(\frac{1}{i} - \frac{1}{n}\right) = \frac{1}{n^{n-1}} \prod_{i=1}^{n-1} \frac{n-i}{i} = \frac{1}{n^{n-1}},$$

so $g_n = 1/n$, and hence $(1 - g_n)^n = (1 - 1/n)^n \to \boxed{1/e}$.

Combinatorics Cancer ♋

23. There are six people living on the same floor of a dorm, where the rooms are numbered $1, 2, \ldots, 6$ (each student lives in a different room). Each pair of residents is either friends or enemies. For every triple of integers (i, j, k) with $1 \le i < j < k \le 6$, the following are true:

    - If the resident in room j is friends with the residents in both room i and room k, then each pair among the three are friends.
    - If the resident in room j is enemies with the residents in both room i and room k, then each pair among the three are enemies.

    How many possible friendship configurations are there among these six people?

    Answer: 720

    Solution: One can show that there is a bijection between friendship configurations and permutations of $\{1, \ldots, 6\}$, where for every pair (i, j) with $i < j$, i and j are friends if and only if i comes before j in the permutation. This gives the answer $6! = \boxed{720}$.

24. Michelle is playing a game with 60 slips of paper, each numbered with a distinct positive integer from 1 to 60. She first lays the slips down in a random order in a line. She can then take two **consecutive** slips and swap them. What is the least number of swaps necessary to guarantee that any initial arrangement can be changed to one where all even numbers are next to at least one other even number and all odd numbers are next to at least one other odd number?

    Answer: 57

    Solution: The idea is that only the parity of the slips matters. We require that in the final arrangement every even number has an even neighbor and every odd number has an odd neighbor, so there are no parity singletons. Consider the arrangement:

    $$1,\ 2, 4, 6, \ldots, 58,\ 3, 5, 7, \ldots, 59,\ 60,$$

    whose parity pattern is $O\,E^{29}\,O^{29}\,E$. At this point the odd number 1 at the left end and the even number 60 at the right end are both isolated.

    To fix the left end, position 2 must contain an odd number. The nearest odd other than 1 is at position 31, so an odd must move left at least $31 - 2 = 29$ positions, requiring at least 29 adjacent swaps. After this, there remain 28 consecutive odds immediately to the left of 60. To fix the right end, position 59 must contain an even number, so an even must cross these 28 odds, requiring at least 28 more adjacent swaps. Thus any solution requires at least $29 + 28 = \boxed{57}$ swaps.

    For the general case, we can proceed by induction. Let n, m be the number of evens and odds present on the slips. If n, m are both even and at least 4, then the number of swaps is at most $n + m - 3$.

    First, we establish a strange base case. Suppose n is 2. Then the worst case is obviously EO^mE, which takes $m = n + m - 2$ swaps to rectify. Now, for a real base case, one can check that when n and m are both equal to 4 that $5 = 4 + 4 - 3$ are sufficient. Now the inductive step. If we have $n, m \ge 4$, there are only two options for the prefix, up to symmetry of swapping evens and odds:

    (a) We start with OO. In this case, we can ignore the starting two letters, which removes two from m; then we know by induction the remaining can be done in $n + (m-2) - 2 = n + m - 4 < n + m - 3$ steps.

    (b) We start with OE. When a block looks like OE^aO, we can move the O a total of a steps to the left. Then, we can remove OO and remove E^a if a is even and E^{a-1} if a is odd. This yields a total number of moves, by induction of at most $a + (n - (a-1)) + (m-2) - 2 = n + m - 3$ moves.

25. Rohit is traveling on the (x, y)-plane, starting at $(0, 0)$, and can only move in the following two ways:

    $$(x, y) \mapsto (x + 3, y + 2), \qquad (x, y) \mapsto (x - 3, y - 1).$$

How many possible sequences of moves allow Rohit to reach the point $(6, 7)$?

Answer: 56

Solution: If Rohit makes a of the first move and b of the second move, to get to $(6, 7)$ we must have that $3a - 3b = 6$ and $2a - b = 7$. This has the solution $(a, b) = (5, 3)$, so any path Rohit makes to $(6, 7)$ must consist of 5 of the first move and 3 of the second. Any ordering of these 8 moves is valid, giving $\binom{8}{3} = \boxed{56}$ possible paths.

26. The 20-dimensional Boolean hypercube H_{20} is a graph G on vertex set $V = \{0, 1\}^{20}$, where vertex $x = (x_1, \ldots, x_{20})$ has an edge to $y = (y_1, \ldots, y_{20})$, if, for exactly a single $i \in \{1, 2, \ldots, 20\}$, $x_i \neq y_i$. Now, vertices are removed from the hypercube H_{20}, disconnecting it, until one connected component has 35 vertices. What is the minimum number of vertices that could have been removed?

Answer: 337

Solution: One can show the set with a fixed number of vertices with the smallest vertex expansion is exactly a d-Hamming ball, that is everything that differs from 0^n in at most d positions, with the minimum possible d. So, the minimal such $1+20 < 35 < \binom{20}{2}+20+1$ sized set is exactly a (subset of a) 2-Hamming ball, so pick $0, e_1, \ldots, e_{20}$, because it needs to also be a superset of a 1-Hamming ball,then $e_1 + e_2, e_1 + e_3, \ldots, e_1 + e_{15}$. To see why the choice of weight-two elements is optimal, consider the graph H where the vertices are 2-sets of [20], and $S_1 \sim S_2$ if $|S_1 \cap S_2| = 1$. Then, we claim that $3|V| - |E|$ of this graph is the amount of weight-3 vertices in the hypercube adjacent to these vertices, because each weight-2 vertex is adjacent to 3 weight-3 ones, except if they have an edge between this means they share a coordinate, and thus have a common neighbor. Therefore, since we must have $H(V) = 14$, the best thing is for H to be a complete graph, which it is under our choice.

So there are $1 + 20 + 14 = 35$ vertices. Now we can count the total neighbors outside the set by their Hamming weight (total number of nonzero entries). The weight-two neighbors are $\binom{20}{2} - 14 = 176$. The weight 3 neighbors are all of the form to $e_1 + e_j + e_k$ where at least one of j or k is in $\{2, \ldots, 15\}$. The number of ways to do this is $\binom{19}{2} - \binom{5}{2} = 161$. In total this yields $176 + 161 = \boxed{337}$ neighbors.

27. Ten people are attending a paintball tournament and will be assigned to two opposing teams of 5: Red Team and Blue Team. (The teams are labeled, so swapping Red and Blue counts as a different team selection.) However, some people are enemies with each other and do not want to be on the same team. For a given set of pairs of enemies E, let $f(E)$ be the number of "valid team selections" with no enemy pairs on the same team. What is the sum of $f(E)$ over all possible sets of enemy pairs?

Answer: $63 \cdot 2^{27}$

Solution: We multiply the probability that two teams with random enemy pairs are valid by the number of possible choices of enemy pairs and choices of teams. Each team has $\binom{5}{2} = 10$ internal pairs that must not be enemies, so there are 20 restrictions which are equally likely to be satisfied as unsatisfied. So the probability of two teams with random enemy pairs being a valid team selection is $\frac{1}{2^{20}}$, while there are 2^{45} possible choices of enemy pairs (45 binary choices). There are $\binom{10}{5}$ choices of teams, by simply choosing the red team and forcing the blue team. Therefore, the answer is $\binom{10}{5} \cdot 2^{25} = \boxed{63 \cdot 2^{27}}$.

28. Let $A_1, A_2, \ldots, A_k \subset \mathbb{R}$ be sets of size at most 2030 such that $\bigcap_{i=1}^{k} A_i = \varnothing$. Suppose there exists an integer $t \geq 1$ such that $|A_i \cap A_j| = t$ for all $1 \leq i < j \leq k$. Calculate the maximum possible value of k.

Answer: $2029^2 + 2029 + 1$

Solution: Note that $p = 2029$ is a prime. We first claim $k \geq p^2 + p + 1$: Indeed, consider $\mathrm{PG}(2, p)$, i.e. the projective plane obtained by projectivizing $\mathbb{F}_p^3$. $\mathrm{PG}(2, p)$ contains $p^2 + p + 1$ points and lines, such that each point is contained in $p + 1$ lines, each line contains $p + 1$ points. Thus no point is contained in every line, and we can embed (the points of) $\mathrm{PG}(2, p)$ in $\mathbb{R}$ and let $A_1, \ldots, A_k$ denote the lines in it.

This paper shows that if $k \geq p^2 + p + 2$, then in fact $A_i \cap A_j = \bigcap_{i=1}^{k} A_i$ for all $1 \leq i < j \leq k$, which would contradict the fact that $\bigcap_{1 \leq i \leq k} A_i = \varnothing$. Here is a partial exposition of the result. Hence our answer is $\boxed{2029^2 + 2029 + 1}$.

Linear Libra ♎

29. Let $V = \mathbb{R}[X]_{\leq 2025}$ be the vector space of real polynomials of degree at most 2025. Consider the derivative map $f : V \to V$, defined by

$$f\left(\sum_{i=0}^{2025} a_i X^i\right) = \sum_{i=1}^{2025} i a_i X^{i-1},$$

where each $a_i \in \mathbb{R}$. Calculate the eigenvalue of f with the largest magnitude.

Answer: 0

Solution: All eigenvalues of f are $\boxed{0}$, since the derivative of any polynomial can't be a scalar multiple of it. Another way to see this is as follows: note that f is nilpotent, i.e. applying f sufficiently many times to any element of R makes it 0. Consequently all eigenvalues of f must be 0.

30. Let $X = \mathbb{C}^{100\times 100}$ and let $T : X \to \mathbb{C}$ be a linear map such that

- $T(AB) = T(BA)$ for every $A, B \in X$, and
- if $C = (c_{jk}) \in X$ is the diagonal matrix with $c_{nn} = n$ for $1 \leq n \leq 100$, then $T(C) = 101$.

Compute $T(M)$, where $M = (m_{jk}) \in X$ is the matrix where $m_{jk} = 1$ if $j + k$ is even and 0 otherwise.

Answer: 2

Solution: Similar to how the determinant is the universal alternating n-linear form on n vectors, the trace is the universal commutator-blind linear functional on a vector space of matrices over a field $\mathbb{F}^{n\times n}$: if $T : \mathbb{F}^{n\times n} \to \mathbb{F}$ is linear and $T(AB) = T(BA)$, then there is a constant $\lambda \in \mathbb{F}$ such that $T(A) = \lambda \operatorname{Tr}(A)$ for all $A \in \mathbb{F}^{n\times n}$. If we apply this theorem directly, we find that

$$101 = T(C) = \lambda \operatorname{Tr}(C) = 5050\lambda,$$

so $\lambda = \frac{1}{50}$. It follows that

$$T(M) = \frac{1}{50}\operatorname{Tr}(M) = \boxed{2}.$$

One can be derive this fact by showing directly that

$$\operatorname{span}\{AB - BA : A, B \in \mathbb{C}^{n\times n}\} = \ker \operatorname{Tr},$$

e.g. by constructing an explicit basis given by matrix units E_{ij} for $i \neq j$ and $E_{ii} - E_{nn}$ for $1 \leq i < n$.

31. Let R be the ring $\mathbb{F}_3[t]/(t^4)$. Let N be the number of 3×3 matrices, $M \in R^{3\times 3}$, such that $\det(M) = 1$ (i.e. the determinant of M is the constant polynomial $1 \in R$). Calculate the number of divisors of N (including 1 and N).

Answer: 280

Solution: We basically have to calculate $|\operatorname{SL}_3(R)|$. Note that $|\operatorname{SL}_3(R)| = |\operatorname{GL}_3(R)|/|R^\times|$. Consider the surjective ring homomorphism $\pi : R \to \mathbb{F}_3$ where every polynomial is mapped to its constant coefficient. Then note that π also induces a surjective ring homomorphism $\tau : \operatorname{GL}_3(R) \to \operatorname{GL}_3(\mathbb{F}_3)$. Consequently, $|\operatorname{GL}_3(R)| = |\operatorname{GL}_3(\mathbb{F}_3)| \cdot |\tau^{-1}(\mathrm{Id})|$. The standard formula for $|\operatorname{GL}_n(\mathbb{F}_p)|$ tells us $|\operatorname{GL}_3(\mathbb{F}_3)| = (3^3 - 1)(3^3 - 3)(3^3 - 3^2)$. $|\tau^{-1}(\mathrm{Id})|$ is just $(3^3)^9$: Indeed, the constant coefficients of the polynomial entries of any matrix in $\tau^{-1}(\mathrm{Id})$ are already specified, and that leaves us with 3 coefficients, each with 3 choices. Finally, $R^\times$ is the set of all polynomials in R with non-zero constant coefficient. Thus $|R^\times| = 2 \times 3^3$. Thus

$$N = |\operatorname{SL}_3(R)| = \frac{26 \times 24 \times 18 \times 3^{27}}{2 \times 3^3} = 2^4 \times 3^{27} \times 13.$$

The number of divisors of N is $(4 + 1) \times (27 + 1) \times (1 + 1) = \boxed{280}$.

32. Compute the largest possible integer m such that there exists $2m$ unit vectors $v_1, v_2, \ldots, v_m$ and $w_1, w_2, \ldots, w_m$ in $\mathbb{R}^{100}$ with the property that for all $1 \le i < j \le m$,

- $v_i \cdot v_j = -1/50$,
- $w_i \cdot w_j = -1/50$,
- $v_i \cdot w_j = -1/100$, and
- $v_i \cdot w_i = 0$.

Answer: 34

Solution: Let Z be the Gram matrix (matrix of inner products) over all the vectors. It can be split up into V, which is the Gram matrix within one group, and X, which represents the inner products between groups. Hence,

$$Z = \begin{pmatrix} V & X \\ X^T & V \end{pmatrix}$$

It is well known that if Z is positive semidefinite, then such a set of vectors exists. Indeed, apply the Cholesky decomposition $Z = M^T M$, then the $2m$ columns of M are what we are in search of. Furthermore, if Z is not positive semidefinite, there can be no such vectors, because then we MUST have $Z = M^T M$ for M being the vectors stacked together. Hence, we are in search of which m make Z PSD.

To do this, we will investigate its eigenvectors. Let J_m be the $m \times m$ all-ones matrix. Then it's clear that $V = (1 + \frac{1}{50})I_m - \frac{1}{50}J_m$ and $X = \frac{1}{100}I_m - \frac{1}{100}J_m$. The all ones vector $\mathbf{1}$ is an eigenvector of J_m (and hence X and V) wherein $J_m\mathbf{1} = m\mathbf{1}$. We thus compute $V\mathbf{1} = \frac{51-m}{50}\mathbf{1}$ and $X\mathbf{1} = \frac{1-m}{100}\mathbf{1}$. We will use ordered pairs to denote two stacked vectors of length m each. This means that $(\mathbf{1}, -\mathbf{1}), (\mathbf{1}, \mathbf{1})$ are eigenvectors of Z. As we can see $Z(\mathbf{1}, -\mathbf{1}) = ((\frac{51-m}{50} - \frac{1-m}{100})\mathbf{1}, (-\frac{51-m}{50} + \frac{1-m}{100})\mathbf{1}) = (\frac{101-m}{100})(\mathbf{1}, -\mathbf{1})$. This eigenvalue is nonnegative if and only if $m \le 101$. Furthermore, $Z(\mathbf{1}, \mathbf{1}) = ((\frac{51-m}{50} + \frac{1-m}{100})\mathbf{1}, \frac{51-m}{50}\frac{1-m}{50})\mathbf{1}) = (\frac{102-3m}{100})(\mathbf{1}, \mathbf{1})$. This writes the constraint $m \le \frac{102}{3} = 34$.

Let $v \perp \mathbf{1}$. Then, $J_m v = 0$ and hence $(v, 0)$ and $(0, v)$ are eigenvectors as for instance $Z(v, 0) = ((1 + \frac{1}{50})v, 0) = (1 + \frac{1}{50})(v, 0)$. Such vectors are clearly orthogonal to the previous eigenvectors. Hence, the remaining $2m - 2$ eigenvalues are all always positive.

Thus, we only have PSDness precisely if and only if $m \le 34$, so our answer is $\boxed{34}$.

33. Calculate the dimension of the vector space S of all 4×4 real symmetric matrices that commute with

$$X = \begin{pmatrix} 1 & -1 & -3 & -1 \\ -1 & 1 & -1 & -3 \\ -3 & -1 & 1 & -1 \\ -1 & -3 & -1 & 1 \end{pmatrix}.$$

Answer: 5

Solution: Since X is symmetric, any symmetric matrix which commutes with X is simultaneously diagonalizable with X. Thus suppose $XY = YX$, and write Y as a matrix in some eigenbasis of X. Note that X is diagonal in this basis, and thus Y has to be block diagonal to commute with X. Now, the eigenvalues of X are $-4, 0, 4, 4$, and thus the dimensions of the blocks are $1, 1, 3$, which makes for a total of $\boxed{5}$.

34. Compute the determinant of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 5 & 4 & 3 \\ 1 & 4 & 25 & 16 & 9 \\ 1 & 8 & 125 & 64 & 27 \\ 1 & 16 & 625 & 256 & 0 \end{pmatrix}.$$

Answer: 5544

Solution: Notice that the matrix above is almost Vandermonde. Thus, one can use the standard Vandermonde determinant, then apply an adjustment to subtract terms that are associated with the bottom right element. Calling the top-left 4×4 matrix as V_4, the whole matrix as A, and

$$V_5 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 5 & 4 & 3 \\ 1 & 4 & 25 & 16 & 9 \\ 1 & 8 & 125 & 64 & 27 \\ 1 & 16 & 625 & 256 & 81 \end{pmatrix}$$

Then we note that $\det A = \det V_5 - 81 \det V_4$. By the standard formula,

$$\det V_5 = (2-1)(3-1)(4-1)(5-1)(3-2)(4-2)(5-2)(4-3)(5-3)(5-4) = 288$$

and

$$\det V_4 = -(2-1)(4-1)(5-1)(4-2)(5-2)(5-4) = -72$$

Thus, the determinant is $\det A = 288 - 81 \cdot (-72) = \boxed{5544}$.

Probability Pisces ♓

35. Cat and David are playing a game. They begin with the number 10 written on a whiteboard. The two then take turns, starting with Cat. On a player's turn, the player chooses a number x uniformly at random from the interval $[0, 1]$, independently of all previous choices, and multiplies the number on the whiteboard by x. They then replace the number on the whiteboard with this product. The game continues until the number on the whiteboard is at most 1. What is the probability that Cat is the last person to take a turn?

Answer: $\frac{101}{200}$

Solution: Let $f(x)$ be the probability that the first player wins, starting from a given x for $x > 1$ (when $x \leq 1$, $f(x) = 1$). Then, with probability $\frac{1}{x}$ the player will immediately win. Else, we want the other player to lose which implies that

$$f(x) = \frac{1}{x} + \int_{\frac{1}{x}}^{1} (1 - f(ux))\,\mathrm{d}u = 1 - \int_{\frac{1}{x}}^{1} f(ux)\,\mathrm{d}u.$$

Consider substituting $y = ux$: then, this transforms into

$$x(1 - f(x)) = \int_1^x f(y)\,\mathrm{d}y.$$

Taking the derivative with respect to x, we have $1 - f(x) - xf'(x) = f(x)$, which gives the differential equation $2f(x) + xf'(x) = 1$. Multiplying through by x (integrating factor), we thus have that $x = 2xf(x) + x^2 f'(x) = (x^2 f(x))'$. Then, $x^2 f(x) = \frac{x^2}{2} + C$, or $f(x) = \frac{1}{2} + \frac{C}{x^2}$. Plugging in $f(1) = 1$, we then have that $f(x) = \frac{1}{2} + \frac{1}{2x^2}$. This implies that the final answer is $\frac{1}{2} + \frac{1}{200} = \boxed{\frac{101}{200}}$.

36. Kaity is playing a number game. She writes the number 324000 on a whiteboard, and at each step she chooses one of the positive divisors of the most recently written number uniformly at random and writes it on the whiteboard. She repeats this process until she writes the number 1. What is the expected value of the natural logarithm of the product of all the numbers written on the whiteboard, including the initial number 324000?

Answer: $10 \ln 2 + 8 \ln 3 + 6 \ln 5$

Solution: We begin by solving the problem in the case that the starting number is p^n. Let P_m denote the desired expected value given that the starting number is m. Then for any positive integer k, we get the recurrence relation

$$P_{p^k} = k \log p + \frac{1}{k+1} \sum_{i=0}^{k} P_{p^i}.$$

Solving the recurrence gives $P_{p^n} = 2n \log p$.

Now, note that $324000 = 2^5 \cdot 3^4 \cdot 5^3$. Since the logarithm of a product is the sum of the logarithms of the individual multiplicands, we have that

$$P_{324000} = P_{2^5} + P_{3^4} + P_{5^3}.$$

By our computation above, we conclude that the desired answer is $\boxed{10 \ln 2 + 8 \ln 3 + 6 \ln 5}$.

37. Let X be a real random variable such that $\mathbb{E}[X^{2k}] = (2k+1)!$ for any positive integer k. Compute the minimum possible value of $\mathbb{P}(X \leq 2026)$.

Answer: $1 - \frac{2027}{e^{2026}}$

Solution: (Thank you to Mehtaab Sawhney for help with this solution.) Let X be any such variable, and let $Y = \varepsilon X$, where ε is 1 with probability $1/2$ and -1 with probability $-1/2$. Then $\mathbb{E}[Y^{2k}] = \mathbb{E}[X^{2k}] = (2k+1)!$ and $\mathbb{E}[Y^{2k+1}] = 0$ for all nonnegative integers k. Hence we can compute the characteristic function of Y to be

$$\varphi_Y(t) = \sum_{k=0}^{\infty} \frac{i^{2k}(2k+1)!t^{2k}}{(2k)!} = \frac{1-t^2}{(1+t^2)^2}.$$

Now, we can compute the probability density function $p_Y(y)$ of Y by taking the Fourier transform:

$$p_Y(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ity} \frac{1-t^2}{(1+t^2)^2} = \frac{1}{2}|y|e^{-|y|},$$

which can be computed via the residue theorem.

It is sufficient to maximize $\mathbb{P}(X > 2026)$. Note that

$$\mathbb{P}(X > 2026) \leq \mathbb{P}(|X| > 2026) = \mathbb{P}(|Y| > 2026)$$

as $|X| = |Y|$. In particular,

$$\mathbb{P}(|Y| > 2026) = \int_{2026}^{\infty} ye^{-y} dy = \frac{2027}{e^{2026}}.$$

Hence $\mathbb{P}(X \leq 2026) \geq \boxed{1 - \frac{2027}{e^{2026}}}$. This is in fact optimal; one can take $X = |Y|$.

38. Let A, B, and C be points chosen independently and uniformly at random on the unit circle centered at O. Let G be the centroid (center of mass) of triangle $\triangle ABC$. What is the expected value of GO^2?

Answer: $\frac{1}{3}$

Solution: Let $\vec{A}$, $\vec{B}$, and $\vec{C}$ be the coordinates for A, B, and C. Then

$$GO^2 = \left\| \frac{\vec{A} + \vec{B} + \vec{C}}{3} \right\|^2.$$

Since $\vec{A}$, $\vec{B}$, and $\vec{C}$ are independent, the expected values of pairwise dot products are 0. Hence

$$\mathbb{E}[GO^2] = \frac{1}{9} \left(\left\|\vec{A}\right\|^2 + \left\|\vec{B}\right\|^2 + \left\|\vec{C}\right\|^2 \right) = \boxed{\frac{1}{3}}.$$

39. Let x, y, z be 3 random elements chosen independently and uniformly from $\{-1, 1\}^n \subset \mathbb{R}^n$. Let p_n be the probability that the triangle formed by x, y, z (as points in $\mathbb{R}^n$) is acute. Calculate

$$\lim_{n\to\infty} \frac{\ln\big(8^n(1-p_n)\big)}{n}.$$

Answer: $\ln 6$

Solution: Note that WLOG we may work on $\{0,1\}^n$ instead of $\{-1,1\}^n$. Note that each point x, y, z can then be thought of as a subset of $[n]$. In particular, 3 points on the boolean hypercube can never form an obtuse (i.e. $> 90°$) angle. Consequently, the triangle formed by x, y, z is not acute iff x, y, z form a right angled triangle. Furthermore, viewed as sets, x, y, z form a right angle at x if $y \cap z \subset x \subset y \cup z$. By symmetry, we may assume $x = \emptyset$. Then the choices we have for y, z are 3^n. Thus the number of triples (x, y, z) forming a right-angled triangle is $= 6^{n+o(1)}$, and thus the answer is $\boxed{\ln 6}$.

40. A cube of side length 2026 is painted on the exterior and is then cut into 2026^3 unit cubes. A unit cube is then chosen uniformly at random and is rolled twice independently. Given that the top face on the first roll is painted, what is the probability that the top face on the second roll is also painted?

Answer: $\frac{1015}{6078}$

Solution: Instead of rolling an individual unit cube, one can imagine cutting the 2026 length cube into 2026^3 unit cubes, picking one of the unit cubes, and gluing all the cubes back together in the same orientation, leaving everything together as one giant cube. Then, one can roll the giant cube, and the side facing up for the unit cube is the face that is rolled. In this case, the condition of the first roll of the unit cube being blue is equivalent the cube being selected being one of the 2026^2 cubes that have a blue face facing up after rolling the giant cube, and all 2026^2 cubes are equally likely to be chosen.

Of these remaining cubes, we then can split them apart and roll our selected unit cube once more. There are $6 \cdot 2026^2$ total possible faces that can roll on top, and $2026^2 + 4 \cdot 2026$ possible blue faces. Each face is equally likely to be rolled, giving a final probability of

$$\frac{2026^2 + 4 \cdot 2026}{6 \cdot 2026^2} = \boxed{\frac{1015}{6078}}.$$